LOOP SPACES AND THE CLASSICAL UNITARY GROUPS

DISSERTATION

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§0. INTRODUCTION

In this paper, we study homological and geometric aspects of function spaces and especially loop spaces related to the classical unitary groups and their homogeneous spaces. Most of the results here are homological, but as a direct consequence, we give a product decomposition for

$$\mathrm{Map}_*(\mathbb{R}P^2, \Omega(SU(4)\langle 3\rangle))$$

where $\mathrm{Map}_*(X,Y)$ denotes the space of pointed maps from X to Y, and $Y\langle j\rangle$ denotes the j-connected cover of Y. It should be indicated that neither $\Omega SU(4)$ nor $\Omega^2 SU(4)$ admits a "non-trivial" product decomposition, although $\Omega^2 SU(4)$ is homotopy equivalent to $S^1 \times \Omega^2(SU(4)\langle 3\rangle)$.

The above decomposition is related to Bott periodicity via the fibration

$$\Omega S^9 \to SU(4)\langle 3\rangle \to SU(5)\langle 3\rangle$$

induced by the classical fibration

$$S^9 \to BSU(4) \to BSU(5) \ .$$

Thus one obtains a fibration

$$\mathrm{Map}_*(\mathbb{R}P^2, \Omega^2 S^9) \to \mathrm{Map}_*(\mathbb{R}P^2, \Omega(SU(4)\langle 3\rangle)) \to \mathrm{Map}_*(\mathbb{R}P^2, \Omega(SU(5)\langle 3\rangle)) \ .$$

Let W_n denote the fibre of the double suspension $E^2\colon S^{2n-1} \to \Omega^2 S^{2n+1}$. We shall prove that there exist spaces X and Y, together with a morphism of fibrations

$$\begin{array}{ccccc}
W_2\times X & \xrightarrow{1\times k} & W_2\times Y & \longrightarrow & \mathrm{Map}_*(\mathbb{R}P^2,\Omega(SU(5)\langle 3\rangle)) \\
\downarrow{\simeq} & & \downarrow{\simeq} & & \downarrow{=} \\
\mathrm{Map}_*(\mathbb{R}P^2,\Omega^2S^9) & \longrightarrow & \mathrm{Map}_*(\mathbb{R}P^2,\Omega(SU(4)\langle 3\rangle)) & \longrightarrow & \mathrm{Map}_*(\mathbb{R}P^2,\Omega(SU(5)\langle 3\rangle))
\end{array}$$

where all spaces are localized at 2 and the vertical maps are homotopy equivalences. Thus one obtains a product decomposition of $\mathrm{Map}_*(\mathbb{R}P^2,\Omega(SU(4)\langle 3\rangle))$ such that one of the factors is both familiar and classical.

An immediate application to homotopy theory is a direct sum decomposition of the mod-2 homotopy groups of S^9 and $SU(4)$. Recall that $\pi_q(X,\mathbb{Z}/p) = [P^q(p),X]_*$ where $P^q(p) = S^{q-1} \cup_p e^q$. From the above decomposition we obtain a commutative diagram where the vertical maps are isomorphisms,

$$\begin{array}{ccc}
\pi_{q-3}W_2 \oplus \pi_{q-3}X & \xrightarrow{1\oplus k_*} & \pi_{q-3}W_2 \oplus \pi_{q-3}Y \\
\downarrow{\simeq} & & \downarrow{\simeq} \\
\pi_q(\Omega S^9;\mathbb{Z}/2) & \longrightarrow & \pi_q(SU(4)\langle 3\rangle;\mathbb{Z}/2)
\end{array}$$

and thus the horizontal map is split on the π_*W_2 summand.

A product decomposition for $\mathrm{Map}_*(\mathbb{R}P^2,\Omega S^5)$ was given in [4], and a related one for $\mathrm{Map}_*(\mathbb{R}P^2,\Omega^2S^9)$ was given in [7]. In particular, $\mathrm{Map}_*(\mathbb{R}P^2,\Omega S^5)$ is homotopy equivalent, at the prime 2, to $W_2\times\Omega^2(S^3\langle 3\rangle)$. This decomposition follows from our results. Also our computations suggest that there may be other decompositions of function spaces involving the special unitary groups, though one may have to consider function spaces whose source is different from $\mathbb{R}P^2$.

The main results here consist of homological computations. We compute the mod-p homology of the double and triple loop spaces of the special unitary groups and their homogeneous spaces. The Bockstein spectral sequences are analyzed to give integral homology. For example, the 2-primary component of the integral homology of $\Omega^2 SU(3)$ consists entirely of $\mathbb{Z}/4$ summands. In addition, the mod-2 homology of the double loop spaces is given as a Hopf-algebra over the Steenrod algebra. Spherical homology classes related to the first unstable element in the homotopy of SU(n) play an important role. Their Hurewicz image in the homology of the iterated loop spaces of SU(n) is given.

§1. STATEMENTS OF MAIN RESULTS

Recall that W_n is the fibre of the double suspension $E^2 \colon S^{2n-1} \to \Omega^2 S^{2n+1}$.

Theorem 1.1. There exist non-trivial spaces X and Y, and a 2-local homotopy commutative diagram,

$$\begin{array}{ccccc} W_2 \times X & \longrightarrow & W_2 \times Y & \longrightarrow & \mathrm{Map}_*(\mathbb{R}P^2, \Omega(SU(5)\langle 3\rangle)) \\ \downarrow \simeq & & \downarrow \simeq & & \downarrow = \\ \mathrm{Map}_*(\mathbb{R}P^2, \Omega^2 S^9) & \longrightarrow & \mathrm{Map}_*(\mathbb{R}P^2, \Omega(SU(4)\langle 3\rangle)) & \longrightarrow & \mathrm{Map}_*(\mathbb{R}P^2, \Omega(SU(5)\langle 3\rangle)) \end{array}$$

where the vertical maps are homotopy equivalences.

That $\mathrm{Map}_*(\mathbb{R}P^2; \Omega^2 S^9)$ is a non-trivial product was proven in [7]. Using Theorem 1.1 we reprove a theorem given in [4] which is a 2-primary analogue of Selick's Theorem [12,13].

Theorem 1.2. Localized at the prime 2, $\mathrm{Map}_*(\mathbb{R}P^2, \Omega S^5)$ is homotopy equivalent to $W_2 \times \Omega^2(S^3\langle 3\rangle)$.

An immediate consequence of Theorem 1.1 is

Corollary 1.3. There is a commutative diagram

$$\begin{array}{ccc} \pi_{q-3} W_2 \oplus \pi_{q-3} X & \xrightarrow{1 \oplus k_*} & \pi_{q-3} W_2 \oplus \pi_{q-3} Y \\ \downarrow \simeq & & \downarrow \simeq \\ \pi_q(\Omega S^9; \mathbb{Z}/2) & \longrightarrow & \pi_q(\Omega(SU(4)\langle 3\rangle); \mathbb{Z}/2) \end{array}$$

where the vertical maps are isomorphisms.

An important ingredient for these product decompositions is the existence of spherical homology classes arising from the first unstable homotopy group of SU(n), which itself is traced to Bott periodicity. To do this, we first give the structure of $H_*(\Omega^2 SU(n);\mathbb{Z}/p)$ and $H_*(\Omega^3(SU(n)\langle 3\rangle);\mathbb{Z}/p)$.

It is convenient to index the algebra generators by the Dyer-Lashof operations and the Bockstein homomorphism. Recall that if X is a (n+1)-fold loop space, there are operations

$$Q_{i(p-1)}: H_q(X;\mathbb{Z}/p) \to H_{pq+i(p-1)}(X;\mathbb{Z}/p)$$

defined for $0\le i\le n$ when n=2 and for $0\le i\le n$, $i\equiv q \bmod 2$ when $p>2$ [6]. The symbol Q_i^a denotes the iterated operation $\underbrace{Q_i \cdots Q_i}_{a\text{-times}}$. We denote the mod-p Bockstein by β.

Remark 1.4. SU(n) is homotopy equivalent to $\Omega BSU(n)$ and $SU(n)\langle 3\rangle$ is homotopy equivalent to $\Omega((BSU(n))\langle 4\rangle)$. We identify SU(n) with $\Omega BSU(n)$ and $SU(n)\langle 3\rangle$ with $\Omega((BSU(n))\langle 4\rangle)$ and shall henceforth consider SU(n) and $SU(n)\langle 3\rangle$ as loop spaces. In particular, this implies that $Q_{2(p-1)}$ and $Q_{3(p-1)}$ are defined in $H_*(\Omega^2 SU(n);\mathbb{Z}/p)$ and $H_*(\Omega^3(SU(n)\langle 3\rangle);\mathbb{Z}/p)$, respectively.

When describing $H_*(\Omega^2 SU(n);\mathbb{Z}/p)$ and $H_*(\Omega^3(SU(n)\langle 3\rangle);\mathbb{Z}/p)$, we give the cases p=2 and p>2 separately. The notation $|x|$ is used to denote the dimension of x. $P[\cdot]$ and $E[\cdot]$ denote polynomial and exterior algebras, respectively. [x] denotes the greatest integer less than or equal to x.

Theorem 1.5. Let n>1. There are choices of primitive elements x_i and $y_{i,0}$ in $H_*(\Omega^2 SU(n);\mathbb{Z}/p)$ where

$$|x_i| = 2i-1 , \quad |y_{i,0}| = 2pi-2 ,$$

and $H_*(\Omega^2 SU(n);\mathbb{Z}/p)$ is isomorphic to one of the following Hopf algebras:

(i) Let p=2.

$$E\left[Q_1^a x_i \mid a\ge 0,\ 1\le i\le \left[\frac{n-1}{2}\right],\ i\not\equiv 0 \bmod 2\right]$$

$$\otimes P\left[Q_1^a x_i \mid a\ge 0,\ \left[\frac{n-1}{2}\right] < i\le n-1,\ i\not\equiv 0 \bmod 2\right]$$

$$\otimes P\left[Q_2^a y_{i,0} \mid a\ge 0,\ \left[\frac{n-1}{2}\right] < i\le n-1,\ i\equiv 0 \bmod 2\right] .$$

(ii) Let p>2.

$$E\left[Q^a_{(p-1)} x_i \mid a\ge 0,\ 1\le i\le n-1,\ i\not\equiv 0 \bmod p\right]$$

$$\otimes P\left[\beta Q^a_{(p-1)} x_i \mid a>0,\ \left[\frac{n-1}{p}\right] < i\le n-1,\ i\not\equiv 0 \bmod p\right]$$

$$\otimes P\left[Q^a_{2(p-1)} y_{i,0} \mid a\ge 0,\ \left[\frac{n-1}{p}\right] < i\le n-1,\ i\equiv 0 \bmod p\right] .$$

The Bockstein spectral sequence for $H_*(\Omega^2 SU(n);\mathbb{Z}/p)$ is analyzed in Proposition 1.18. A complete description of the Dyer-Lashof operations and the action of the Steenrod algebra in $H_*(\Omega^2 SU(n);\mathbb{Z}/2)$ is given in Propositions 1.12 through 1.14.

Recall that if $n \le p$, then localized at p, SU(n) is homotopy equivalent to $S^3 \times \cdots \times S^{2n-1}$ [11,15]. Thus

$$H_*(\Omega^3(SU(n)\langle 3\rangle);\mathbb{Z}/p)$$
$$\simeq H_*(\Omega_0^3 S^3;\mathbb{Z}/p) \otimes H_*(\Omega^3 S^5;\mathbb{Z}/p) \otimes \cdots \otimes H_*(\Omega^3 S^{2n-1};\mathbb{Z}/p) .$$

For $n>p$ we have the following theorem.

<u>Theorem 1.6.</u> Let $n>p$. There are choices of elements u, u_i and $v_{i,0}$ in $H_*(\Omega^3(SU(n)\langle 3\rangle);\mathbb{Z}/p)$ where

$$|u| = 2p-2 , \qquad |u_i| = 2i-2 , \qquad |v_{i,0}| = 2pi-3 ,$$

and $H_*(\Omega^3(SU(n)\langle 3\rangle);\mathbb{Z}/p)$ is isomorphic to one of the following algebras:

(i) Let $p=2$.

$$P\left[Q_2^a u \mid a\ge 0\right] \otimes P\left[Q_2^a u_i \mid a\ge 0,\ 1<i\le \left[\tfrac{n-1}{2}\right],\ i\not\equiv 0 \bmod 2\right]$$
$$\otimes P\left[Q_1^a Q_2^b u_i \mid a,b\ge 0,\ \left[\tfrac{n-1}{2}\right] < i \le n-1,\ i\not\equiv 0 \bmod 2\right]$$
$$\otimes P\left[Q_1^a Q_3^b v_{i,0} \mid a,b\ge 0,\ \left[\tfrac{n-1}{2}\right] < i \le n-1,\ i\equiv 0 \bmod 2\right]$$

(ii) Let $p>2$.

$$P\left[Q^a_{2(p-1)}u \mid a\geq 0\right] \otimes P\left[Q^a_{2(p-1)}u_i \mid a\geq 0,\ 1<i\leq n-1,\ i\not\equiv 0 \bmod p\right]$$
$$\otimes E\left[Q^a_{(p-1)}\beta Q^b_{2(p-1)}u_i \mid a\geq 0,\ b>0,\ \left[\frac{n-1}{p}\right] < i \leq n-1,\ i\not\equiv 0 \bmod p\right]$$
$$\otimes P\left[\beta Q^a_{(p-1)}\beta Q^b_{2(p-1)}u_i \mid a,b>0,\ \left[\frac{n-1}{p}\right] < i \leq n-1,\ i\not\equiv 0 \bmod p\right]$$
$$\otimes E\left[Q^a_{(p-1)}Q^b_{3(p-1)}v_{i,0} \mid a,b\geq 0,\ \left[\frac{n-1}{p}\right] < i \leq n-1,\ i\equiv 0 \bmod p\right]$$
$$\otimes P\left[\beta Q^a_{(p-1)}Q^b_{3(p-1)}v_{i,0} \mid a>0,\ b\geq 0,\ \left[\frac{n-1}{p}\right] < i \leq n-1,\ i\equiv 0 \bmod p\right]$$

Remark 1.7. $H_*(\Omega^3(SU(n)\langle 3\rangle);\mathbb{Z}/p)$ is not generally primitively generated. In particular, the generators of the form u_{kp+1} cannot be chosen to be primitive. Thus for $n>p+1$, $H_*(\Omega^3(SU(n)\langle 3\rangle);\mathbb{Z}/p)$ is not primitively generated; however, $H_*(\Omega^3(SU(p+1)\langle 3\rangle);\mathbb{Z}/p)$ is primitively generated. If $n<p+1$, then $H_*(\Omega^3(SU(n)\langle 3\rangle);\mathbb{Z}/p)$ is primitively generated if and only if $p\neq 2$.

Let $\phi\colon \pi_q X \to H_q(X;\mathbb{Z})$ denote the Hurewicz homomorphism, and $\rho\colon H_q(X;\mathbb{Z}) \to H_q(X;\mathbb{Z}/p)$ the mod-p reduction map. Let Φ denote the composite $\rho\circ\phi$. In the next theorem, we give information on the image of Φ for $\Omega^2SU(n)$ and $\Omega^3(SU(n)\langle 3\rangle)$.

Theorem 1.8. The following elements are in the image of Φ.

(i) If $n\not\equiv 0 \bmod p$, then $\beta Q_{2(p-1)}u_n \in H_{2pn-3}(\Omega^3(SU(pn)\langle 3\rangle;\mathbb{Z}/p)$ and thus, by suspending, $\beta Q_{(p-1)}x_n \in H_{2pn-2}(\Omega^2SU(pn);\mathbb{Z}/p)$ are in Im Φ.

(ii) If $n \equiv 0 \bmod p$, then $v_{n,0} \in H_{2pn-3}(\Omega^3(SU(pn)\langle 3\rangle);\mathbb{Z}/p)$ and thus, by suspending, $y_{n,0} \in H_{2pn-2}(\Omega^2 SU(pn);\mathbb{Z}/p)$ are in Im Φ.

Theorem 1.8 plays a crucial role in our decomposition theorems above and our homology computations. A related theorem is given by the following.

Theorem 1.9. Let $n \equiv j-1 \bmod p$ for $1 \le j \le p$. If $2n > k \ge 2j$, then there exists a non-zero element in $H_{2n-k}(\Omega^k(SU(n)\langle k\rangle);\mathbb{Z}/p)$ which is in Im Φ.

The first unstable homotopy group of SU(n) is $\pi_{2n}SU(n) = \mathbb{Z}/n!$ [16]. Let $\mu: S^{2n} \to SU(n)$ be a generator of this group and denote by μ^j the adjoint of μ, $\mu^j: S^{2n-j} \to \Omega^j SU(n)$. Theorem 1.9 gives a partial answer to the question of how large we must choose j in order for μ^j to have a non-trivial image under Φ. In particular, if $n \equiv j-1 \bmod p$ where $1 \le j \le p$, then μ^{2j} has non-trivial image under Φ. In the cases of $n \equiv 0$ or $1 \bmod p$, this answer is best possible.

Next we describe the mod-p homology of the double and triple loops of the homogeneous spaces SU(n)/SU(m). The following sets of integers are used as indexing sets:

$$A_p(n,m) = \left\{ i \in \mathbb{Z} \;\middle|\; \begin{array}{l} m \le i \le \left\lceil \frac{n-1}{p} \right\rceil \text{ and } i \equiv 0 \bmod p \quad \text{or} \\ m \le i \le \min\left\{ p(m-1), \left\lceil \frac{n-1}{p} \right\rceil \right\} \text{ and} \\ \hfill i \equiv 0 \bmod p \end{array} \right\}$$

$$B_p(n,m) = \left\{ i \in \mathbb{Z} \;\middle|\; \begin{array}{l} \max\left\{ m-1, \left\lceil \frac{n-1}{p} \right\rceil \right\} < i \le n-1 \text{ and} \\ \hfill i \not\equiv 0 \bmod p \text{ or} \\ \max\left\{ m-1, \left\lceil \frac{n-1}{p} \right\rceil \right\} < i \le \min\{p(m-1), n-1\} \\ \hfill \text{and } i \equiv 0 \bmod p \end{array} \right\}$$

$$C_p(n,m) = \left\{ i \in \mathbb{Z} \;\middle|\; \begin{array}{l} \max\left\{ p(m-1), \left\lceil \frac{n-1}{p} \right\rceil \right\} < i \le n-1 \text{ and} \\ \hfill i \equiv 0 \bmod p \end{array} \right\}$$

Theorem 1.10. Let $1 \le m < n$. There are choices of primitive elements x_i and $y_{i,a}$ in $H_*(\Omega^2(SU(n)/SU(m));\mathbb{Z}/p)$ where

$$|x_i| = 2i-1 , \qquad |y_{i,a}| = 2p^{a+1}i - 2 ,$$

and $H_*(\Omega^2(SU(n)/SU(m));\mathbb{Z}/p)$ is isomorphic to one of the following Hopf algebras:

(i) Let p=2.

$$E[Q_1^a x_i \mid a \ge 0,\ i \in A_2(n,m)]$$
$$\otimes P[Q_1^a x_i \mid a \ge 0,\ i \in B_2(n,m)]$$
$$\otimes P[y_{i,a} \mid a \ge 0,\ i \in C_2(n,m)]$$

(ii) Let $p>2$.

$$E[Q^{a}_{(p-1)}x_i \mid a\geq 0,\ i\in A_p(n,m)\cup B_p(n,m)]$$

$$\otimes P[\beta Q^{a}_{(p-1)}x_i \mid a>0,\ i\in B_p(n,m)]$$

$$\otimes P[y_{i,a} \mid a\geq 0,\ i\in C_p(n,m)]$$

<u>Theorem 1.11.</u> Let $2\leq m<n$. There are choices of elements u_i and $v_{i,a}$ in $H_*(\Omega^3(SU(n)/SU(m));\mathbb{Z}/p)$ where

$$|u_i| = 2i-2\ , \qquad |v_{i,a}| = 2p^{a+1}i - 3\ ,$$

and $H_*(\Omega^3(SU(n)/SU(m));\mathbb{Z}/p)$ is isomorphic to one of the following algebras:

(i) Let $p=2$.

$$P[Q^{a}_{2}u_i \mid a\geq 0,\ i\in A_p(n,m)]\otimes P[Q^{a}_{1}Q^{b}_{2}u_i \mid a,b\geq 0,\ i\in B_p(n,m)]$$

$$\otimes P[Q^{a}_{1}v_{i,b} \mid a,b\geq 0,\ i\in C_p(n,m)]$$

(ii) Let $p>2$.

$$P[Q^{a}_{2(p-1)}u_i \mid a\geq 0,\ i\in A_p(n,m)\cup B_p(n,m)]$$

$$\otimes E[Q^{a}_{(p-1)}\beta Q^{b}_{2(p-1)}u_i \mid a\geq 0,\ b>0,\ i\in B_p(n,m)]$$

$$\otimes P[\beta Q^{a}_{(p-1)}\beta Q^{b}_{2(p-1)}u_i \mid a,b>0,\ i\in B_p(n,m)]$$

$$\otimes E[Q^{a}_{(p-1)}v_{i,b} \mid a,b\geq 0,\ i\in C_p(n,m)]$$

$$\otimes P[\beta Q^{a}_{(p-1)}v_{i,b} \mid a>0,\ b\geq 0,\ i\in C_p(n,m)]$$

A partial description of the Dyer-Lashof operations in $H_*(\Omega^2 SU(n);\mathbb{Z}/2)$ and $H_*(\Omega^2(SU(n)/SU(m));\mathbb{Z}/2)$ was given in Theorems 1.5 and 1.10. The following two propositions, together with the Adem relations, complete this description.

Proposition 1.12. Let $n>1$. Then in $H_*(\Omega^2 SU(n);\mathbb{Z}/2)$

$$Q_2 x_i = 0 = Q_1 Q_2^a y_{i,0} .$$

Let $n>m\geq 1$. Then in $H_*(\Omega^2(SU(n)/SU(m));\mathbb{Z}/2)$

$$Q_1 y_{i,a} = 0 .$$

Proposition 1.13. Let $n>m\geq 1$. Then the Browder operation λ_1 in $H_*(\Omega^2(SU(n)/SU(m));\mathbb{Z}/2)$ and λ_2 in $H_*(\Omega^2 SU(n);\mathbb{Z}/2)$ are trivial.

Next we consider the action of the Steenrod algebra in $H_*(\Omega^2 SU(n);\mathbb{Z}/2)$.

Proposition 1.14. The action of the Steenrod algebra on $H_*(\Omega^2 SU(n);\mathbb{Z}/2)$ is determined by the Nishida relations and the following formulas.

(i) $Sq_*^{2j+1} x_i = 0$

$Sq_*^{2j} x_i = (i-2j-1,j) Q_1^a x_k$ where $i-j = 2^a k$, $k \not\equiv 0 \bmod 2$.

(ii) $Sq_*^j y_{i,0} = 0$ if $j \not\equiv 0 \bmod 4$

$$Sq_*^{4j} y_{i,0} = \begin{cases} (i-2j,j)\, y_{i-j,0} & i-j \equiv 0 \bmod 2,\ i-j > \left[\frac{n-1}{2}\right] \\ (i-2j,j)\,(x_{i-j})^2 & i-j \not\equiv 0 \bmod 2,\ i-j > \left[\frac{n-1}{2}\right] \\ 0 & \text{otherwise}. \end{cases}$$

Proposition 1.15. Let $j: SU(n) \to SU(n)/SU(m)$ be the natural quotient map. Then $\Omega^2 j_*: H_*(\Omega^2 SU(n);\mathbb{Z}/2) \to H_*(\Omega^2(SU(n)/SU(m));\mathbb{Z}/2)$ is given by the following formulas.

(i) $$\Omega^2 j_*(x_i) = \begin{cases} x_i & i \geq m \\ 0 & i < m \end{cases}$$

(ii) $$\Omega^2 j_*(Q_2^a y_{i,0}) = \begin{cases} y_{i,a} & 2(m-1) < i \\ (Q_1^a x_i)^2 & m-1 < i \leq 2(m-1) \\ 0 & i \leq m-1 . \end{cases}$$

Note that the only generators of $H_*(\Omega^2(SU(n)/SU(m));\mathbb{Z}/2)$ not in the image of $\Omega^2 j_*$ are those of the form $Q_1^a x_i$ for $i \equiv 0 \bmod 2$. Thus to compute the action of the Steenrod algebra on $H_*(\Omega^2(SU(n)/SU(m));\mathbb{Z}/2)$, it suffices to compute the Steenrod operations on the element x_i where $i \equiv 0 \bmod 2$.

Proposition 1.16. Consider $x_i \in H_*(\Omega^2(SU(n)/SU(m));\mathbb{Z}/2)$ where $i \equiv 0 \bmod 2$. Then

$$Sq_*^{2j+1} x_i = 0$$

$$Sq_*^{2j} x_i = \begin{cases} (i-2j,j)\, x_{i-j} & i-j \geq m \\ 0 & i-j < m . \end{cases}$$

The action of the higher Bocksteins is given in the following proposition.

Proposition 1.17. Let $1 \le m < n$. Let $s_n(i) = s$ be such that $p^{s-1} i < n \le p^s i$. The differentials in the Bockstein spectral sequence for $H_*(\Omega^2(SU(n)/SU(m));\mathbb{Z}/p)$ are given by:

(i) Let $i \in A_p(n,m) \cup B_p(n,m)$ and $0 \le a < s_n(i)$. Then $\beta^r(Q^a_{(p-1)} x_i) = 0$ for all r.

(ii) Let $i \in B_p(n,m)$ and $a \ge 1$. Then
$\beta^1(Q^a_{(p-1)} x_i) = \beta Q^a_{(p-1)} x_i$.

(iii) Let $i \in A_p(n,m)$ and $a \ge s_n(i)$. Then $\beta^r(Q^a_{(p-1)} x_i) = 0$ for $r < s_n(i)$. Thus β^s is defined and
$\beta^s(Q^a_{(p-1)} x_i) = y_{p^{s-1} i, a-s}$.

From the action of the higher Bocksteins in the mod-p homology of $\Omega^2 SU(n)$, we obtain the following corollary concerning the p-torsion in $H_*(\Omega^2(SU(n)/SU(m));\mathbb{Z})$.

Corollary 1.18. Let s and n be such that $p^{s-1} m < n \le p^s m$. Then p^s annihilates the p-torsion in $H_*(\Omega^2(SU(n)/SU(m));\mathbb{Z})$, but p^{s-1} does not.

Example 1.19. The 2-torsion in $H_*(\Omega^2 SU(3);\mathbb{Z})$ is a direct sum of $\mathbb{Z}/4$-summands. The 2-torsion in $H_*(\Omega^2 SU(4);\mathbb{Z})$ is a direct sum of $\mathbb{Z}/2$- and $\mathbb{Z}/8$-summands.

§2. PRODUCT DECOMPOSITION OF FUNCTION SPACES AND PROOFS OF STATEMENTS 1.1-1.3

In this section, all spaces are localized at the prime 2. To prove statements 1.1-1.3, we shall construct maps $h\colon \Omega^2(S^3\langle 3\rangle) \to \mathrm{Map}_*(\mathbb{R}P^2,\Omega S^5)$ and $f\colon W_2 \to \mathrm{Map}_*(\mathbb{R}P^2,\Omega S^5)$ with the following properties. h_* is non-zero on $H_2(-,\mathbb{Z}/2)$ and f_* is non-zero on $H_5(-;\mathbb{Z}/2)$. Denote by γ the composite

$$\Omega^2(S^3\langle 3\rangle)\times W_2 \xrightarrow{h\times f} \mathrm{Map}_*(\mathbb{R}P^2,\Omega S^5)\times \mathrm{Map}_*(\mathbb{R}P^2,\Omega S^5)$$

$$\xrightarrow{\text{multiply}} \mathrm{Map}_*(\mathbb{R}P^2,\Omega S^5) \ .$$

Theorem 1.2 follows at once from the existence of these maps, together with the following two lemmas.

Lemma 2.1. Let $f\colon X \to \mathrm{Map}_*(\mathbb{R}P^2,\Omega S^{2n+1})$ be a map which induces a mod-2 homology isomorphism on the module of primitives in dimension 2n-2 and 4n-3. If the mod-2 homology of X is isomorphic to that of $\mathrm{Map}_*(\mathbb{R}P^2,\Omega S^{2n+1})$ as a coalgebra over the Steenrod algebra, then f_* is an isomorphism.

Lemma 2.2. (i) The mod-2 homology of $\Omega^2 S^3\langle 3\rangle\times W_2$ is isomorphic to that of $\mathrm{Map}_*(\mathbb{R}P^2,\Omega S^5)$ as a coalgebra over the Steenrod algebra.

(ii) γ induces an isomorphism on the module of primitives in dimensions 2 and 5.

Lemma 2.1 is proved in [3] and Lemma 2.2 is proved in [4]. Next we construct the maps h and f.

Consider the cofibration

$$S^1 \xrightarrow{[2]} S^1 \longrightarrow \mathbb{R}P^2$$

where [2] is the degree 2 map. By applying $Map_*(-;X)$ to this cofibration we obtain the fibration

$$\Omega X \xleftarrow{2} \Omega X \longleftarrow Map_*(\mathbb{R}P^2, X)$$

where 2 is the H-space squaring map, [14, p.97]. Let $[A,B]$ denote homotopy classes of maps $f: A \to B$. If $f \in [Y, \Omega X]$ has order 2, then the composite

$$\begin{array}{ccc} Y & \xrightarrow{f} & \Omega X \\ & & \downarrow 2 \\ & & \Omega X \end{array}$$

is null. Thus f lifts to $Map_*(\mathbb{R}P^2, X)$.

Consider the Hilton-Hopf invariant $H_2: \Omega S^{2n+1} \to \Omega S^{4n+1}$. The following lemma is useful in the construction of both h and f. A proof of this result is given in [5].

<u>Lemma 2.3.</u> ΩH_2 has order 2 in the abelian group $[\Omega^2 S^{2n+1}, \Omega^2 S^{4n+1}]$ and thus there is a lift of ΩH_2 to $Map_*(\mathbb{R}P^2, \Omega S^{4n+1})$. Furthermore, this lift induces an isomorphism on $H_{4n-2}(-;\mathbb{Z}/2)$.

<u>CONSTRUCTION OF h:</u> By Lemma 2.4, there is a lift of $\Omega H_2\colon \Omega^2 S^3 \to \Omega^2 S^5$ to $\mathrm{Map}_*(\mathbb{R}P^2, \Omega S^5)$. Let $h\colon \Omega^2 S^3\langle 3\rangle \to \mathrm{Map}_*(\mathbb{R}P^2, \Omega S^5)$ be any choice of this lift restricted to $\Omega^2 S^3\langle 3\rangle$. By Lemma 2.4, h_* induces an isomorphism on $H_2(-;\mathbb{Z}/2)$.

<u>CONSTRUCTION OF f:</u> The map f is defined as the composite

$$\begin{array}{ccc}
W_2 & \xrightarrow{\quad f \quad} & \mathrm{Map}_*(\mathbb{R}P^2, \Omega S^5) \\
\Big\downarrow{\scriptstyle \sigma_2} & & \Big\uparrow{\scriptstyle \zeta} \\
\mathrm{Map}_*(\mathbb{R}P^2, \Omega^2 S^9) & \xrightarrow{\quad \xi_4 \quad} & \mathrm{Map}_*(\mathbb{R}P^2, \Omega(SU(4)\langle 3\rangle))
\end{array}$$

where σ_2 is defined in [4] and also below.

Consider the following diagram

$$\begin{array}{ccc}
W_n & \overset{\sigma_n}{\dashrightarrow} & \mathrm{Map}_*(\mathbb{R}P^2, \Omega^2 S^{4n+1}) \\
\Big\downarrow & & \Big\downarrow \\
S^{2n-1} & \longrightarrow & * \\
\Big\downarrow{\scriptstyle E^2} & & \Big\downarrow \\
\Omega^2 S^{2n+1} & \xrightarrow{\quad \bar{h} \quad} & \mathrm{Map}_*(\mathbb{R}P^2, \Omega S^{4n+1})
\end{array}$$

where E^2 is the double suspension and $\bar{h}$ is a choice of lift of $\Omega H_2\colon \Omega^2 S^{2n+1} \to \Omega^2 S^{4n+1}$ as in Lemma 2.4. The columns are fibrations and the solid square homotopy commutes since $\pi_{2n-1}\, \mathrm{Map}_*(\mathbb{R}P^2, \Omega S^{4n+1}) = 0$. Thus there exists a choice of map σ_n making the above diagram a morphism of fibrations.

<u>Lemma 2.4.</u> The map $(\sigma_n)_*$ is an epimorphism on $H_{4n-3}(-;\mathbb{Z}/2)$.

<u>Proof:</u> By the long exact sequence in homotopy for the fibration $2: \Omega^2 S^{4n+1} \to \Omega^2 S^{4n+1}$, $\mathrm{Map}_*(\mathbb{R}P^2, \Omega S^{4n+1})$ is $(4n-3)$-connected and $\pi_{4n-2}\, \mathrm{Map}_*(\mathbb{R}P^2, \Omega S^{4n+1}) = \mathbb{Z}/2$. Thus $H_{4n-2}(\mathrm{Map}_*(\mathbb{R}P^2, \Omega S^{4n+1});\mathbb{Z}/2) = \mathbb{Z}/2$. In the Serre spectral sequence for the path space fibration of $\mathrm{Map}_*(\mathbb{R}P^2, \Omega S^{4n+1})$, the generator of $H_{4n-2}(\mathrm{Map}_*(\mathbb{R}P^2, \Omega S^{4n+1});\mathbb{Z}/2)$ must transgress to the generator of $H_{4n-3}(\mathrm{Map}_*(\mathbb{R}P^2, \Omega^2 S^{4n+1});\mathbb{Z}/2)$. Since $\bar{h}_*$ is non-zero on $H_{4n-2}(-;\mathbb{Z}/2)$, there is an element y in $H_{4n-2}(\Omega^2 S^{2n+1};\mathbb{Z}/2)$ whose image under $\bar{h}_*$ is the generator of $H_{4n-2}(\mathrm{Map}_*(\mathbb{R}P^2, \Omega S^{4n+1});\mathbb{Z}/2)$. By naturality, y must transgress to an element in $H_{4n-3}(W_n;\mathbb{Z}/2)$ whose image under $(\sigma_n)_*$ is the generator of $H_{4n-3}(\mathrm{Map}_*(\mathbb{R}P^2, \Omega^2 S^{4n+1});\mathbb{Z}/2)$. Thus $(\sigma_n)_*$ is an epimorphism on $H_{4n-3}(-;\mathbb{Z}/2)$. □

Next we define ξ_n: $\mathrm{Map}_*(\mathbb{R}P^2, \Omega^2 S^{2n+1}) \to \mathrm{Map}_*(\mathbb{R}P^2, \Omega(SU(n)\langle 3\rangle))$. Consider the classical fibration

$$S^{2n+1} \xrightarrow{\alpha_n} BSU(n) \longrightarrow BSU(n+1) \quad .$$

The map α_n is used to define ξ_n and has many applications in subsequent sections. For $j \le 2n$, let α_n^j: $S^{2n+1} \to (BSU(n))\langle j\rangle$ be a lift of α_n to j-connected covers. For j in this range, it will cause no confusion to denote α_n^j simply by α_n. By the long exact sequence in homotopy for the above fibration and the fact that

$\pi_{2n+1}BSU(n+1) = 0$, α_n is a generator of $\pi_{2n+1}BSU(n)$. The map ξ_n is the map obtained by applying $Map_*(\mathbb{R}P^2,-)$ to

$$\Omega^2 S^{2n+1} \xrightarrow{\Omega^2\alpha_n} \Omega(SU(n)\langle 3\rangle) .$$

To define $\zeta\colon Map_*(\mathbb{R}P^2;\Omega(SU(4)\langle 3\rangle)) \to Map_*(\mathbb{R}P^2,\Omega S^5)$, note there is a strictly commutative diagram

$$\begin{array}{ccccc}
SU(2) & \longrightarrow & SU(3) & \longrightarrow & S^5 \\
\downarrow & & \downarrow & & \downarrow \\
Sp(2) & \longrightarrow & SU(4) & \xrightarrow{g} & SU(4)/Sp(2) \\
\downarrow & & \downarrow & & \downarrow \\
S^7 & \longrightarrow & S^7 & \longrightarrow & *
\end{array}$$

where all the rows and columns are strict fibrations and $Sp(n)$ denotes the symplectic group. Thus $SU(4)/Sp(2)$ is homeomorphic to S^5. Denote the restriction of g to $SU(4)\langle 3\rangle$ by $\bar{g}$. Then ζ is defined to be the map obtained by applying $Map_*(\mathbb{R}P^2,-)$ to

$$\Omega(SU(4)\langle 3\rangle) \xrightarrow{\Omega\bar{g}} \Omega S^5 .$$

We will show that $(\zeta\circ\xi_4)_*$ is non-zero on $H_5(-;\mathbb{Z}/2)$. The following two lemmas are used in the proof of this fact.

<u>Lemma 2.5.</u> Consider the natural maps $i\colon SU(3)\langle 3\rangle \to SU(4)\langle 3\rangle$ and $p\colon SU(3)\langle 3\rangle \to S^5$. Then

(i) $\Omega^3 i_*(v_{2,0}) = v_{2,0}$;

(ii) $\Omega^3 p_*(v_{2,0}) \neq 0$.

Lemma 2.6. Let $\lambda: \Omega^3 S^5 \to \mathrm{Map}_*(\mathbb{R}P^2, \Omega S^5)$ be a choice map induced by the fibration

$$\mathrm{Map}_*(\mathbb{R}P^2, \Omega S^5) \longrightarrow \Omega^2 S^5 \xrightarrow{\ 2\ } \Omega^2 S^5 .$$

Then λ_* is a monomorphism.

Lemma 2.5 follows from more general results proved in Section 5 and Lemma 2.6 is proved in [3].

Lemma 2.7. The map $(\zeta \circ \xi_4)_*$ is non-zero on $H_5(-;\mathbb{Z}/2)$.

Proof: Consider the homotopy commutative diagram

$$\begin{array}{ccc} \Omega^4 S^9 & \xrightarrow{\ \Omega^3 \bar{g} \circ \Omega^4 \alpha_4\ } & \Omega^3 S^5 \\ \downarrow & & \downarrow \lambda \\ \mathrm{Map}_*(\mathbb{R}P^2; \Omega^2 S^9) & \xrightarrow{\ \zeta \circ \xi_4\ } & \mathrm{Map}_*(\mathbb{R}P^2, \Omega S^5) \end{array} .$$

By Lemma 2.6, λ_* is a monomorphism. Thus it suffices to show that $(\Omega^3 \bar{g} \circ \Omega^4 \alpha_4)_*$ is non-zero on $H_5(-;\mathbb{Z}/2)$. By Theorem 1.6,

$$H_*(\Omega^3(SU(3)\langle 3\rangle);\mathbb{Z}/2) = P[Q_2^a u \mid a \geq 0] \otimes P[Q_1^a Q_3^b v_{2,0} \mid a,b \geq 0]$$

and

$$H_*(\Omega^3(SU(4)\langle 3\rangle);\mathbb{Z}/2)$$

$$= P[Q_2^a u \mid a \geq 0] \otimes P[Q_1^a Q_3^b v_{2,0} \mid a,b \geq 0] \otimes P[Q_1^a Q_2^b u_3 \mid a,b \geq 0] .$$

By Lemma 2.5, $\Omega^3 i_*(v_{2,0}) = v_{2,0}$ where $i: SU(3)\langle 3\rangle \to SU(4)\langle 3\rangle$ and $\Omega^3 p_*(v_{2,0}) \neq 0$ where $p: SU(3)\langle 3\rangle \to S^5$. Thus $\Omega^3 \bar{g}_*(v_{2,0}) \neq 0$.

By Theorem 1.8, the image of $(\Omega^4\alpha_4)_*$ is generated by $v_{2,0}$. This implies that $(\Omega^3\bar{g}\circ\Omega^4\alpha_4)_*$ is non-zero on $H_5(-;\mathbb{Z}/2)$. □

By Lemmas 2.4 and 2.7, f_* is non-zero on $H_5(-;\mathbb{Z}/2)$. This completes the construction of f.

Having constructed h and f with the required properties, we assume that the map

$$\gamma\colon\ \Omega^2(S^3\langle 3\rangle)\times W_2 \xrightarrow{\ \gamma\ } \mathrm{Map}_*(\mathbb{RP}^2,\Omega S^5)$$

is a homotopy equivalence. The proof of Theorem 1.1 is given next.

<u>Proof of Theorem 1.1:</u> Since the map $\xi_4\colon \mathrm{Map}_*(\mathbb{RP}^2,\Omega^2S^9)\to \mathrm{Map}_*(\mathbb{RP}^2,\Omega(SU(4)\langle 3\rangle))$ in the factorization of f arises from the fibration

$$\Omega^2S^9 \xrightarrow{\ \Omega^2\alpha_4\ } \Omega(SU(4)\langle 3\rangle) \longrightarrow \Omega(SU(5)\langle 3\rangle)\ ,$$

there is a homotopy commutative diagram where the rows and columns are fibrations.

$$\begin{array}{ccccc}
X & \xrightarrow{\ k\ } & Y & \longrightarrow & \mathrm{Map}_*(\mathbb{RP}^2,\Omega(SU(5)\langle 3\rangle)) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Map}_*(\mathbb{RP}^2,\Omega^2S^9) & \xrightarrow{\ \xi_4\ } & \mathrm{Map}_*(\mathbb{RP}^2,\Omega(SU(4)\langle 3\rangle)) & \longrightarrow & \mathrm{Map}_*(\mathbb{RP}^2,\Omega(SU(5)\langle 3\rangle)) \\
\downarrow{\scriptstyle \theta_1} & & \downarrow{\scriptstyle \theta_2} & & \downarrow \\
W_2 & \xrightarrow{\ 1\ } & W_2 & \longrightarrow & *
\end{array}$$

The maps θ_1 and θ_2 are defined by the commutativity of the following diagram:

$$\begin{array}{ccccc}
\mathrm{Map}_*(\mathbb{RP}^2,\Omega^2S^9) & \xrightarrow{\xi_4} & \mathrm{Map}_*(\mathbb{RP}^2,\Omega(SU(4)\langle 3\rangle)) & \xrightarrow{\zeta} & \mathrm{Map}_*(\mathbb{RP}^2,\Omega S^5) \\
& \searrow^{\theta_1} & & \searrow^{\theta_2} & \big\downarrow \text{projection} \\
& & & & W_2
\end{array}$$

X is the homotopy fibre of θ_1 and Y is the homotopy fibre of θ_2. Since θ_1 and θ_2 have a cross-section given by

$$W_2 \xrightarrow{\sigma_2} \mathrm{Map}_*(\mathbb{RP}^2,\Omega S^9) \xrightarrow{\xi_4} \mathrm{Map}_*(\mathbb{RP}^2,\Omega(SU(4)\langle 3\rangle)) ,$$

there is a homotopy commutative diagram

$$\begin{array}{ccccc}
W_2\times X & \longrightarrow & W_2\times Y & \longrightarrow & *\times\mathrm{Map}_*(\mathbb{RP}^2,\Omega(SU(5)\langle 3\rangle)) \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
\mathrm{Map}_*(\mathbb{RP}^2,\Omega^2S^9)^2 & \longrightarrow & \mathrm{Map}_*(\mathbb{RP}^2,\Omega(SU(4)\langle 3\rangle))^2 & \longrightarrow & \mathrm{Map}_*(\mathbb{RP}^2,\Omega(SU(5)\langle 3\rangle))^2 \\
\big\downarrow \text{multiply} & & \big\downarrow \text{multiply} & & \big\downarrow \text{multiply} \\
\mathrm{Map}_*(\mathbb{RP}^2,\Omega^2S^9) & \longrightarrow & \mathrm{Map}_*(\mathbb{RP}^2,\Omega(SU(4)\langle 3\rangle)) & \longrightarrow & \mathrm{Map}_*(\mathbb{RP}^2,\Omega(SU(5)\langle 3\rangle))
\end{array}$$

where the rows are fibrations and the vertical composites are isomorphisms in homotopy. Thus the vertical composites are homotopy equivalences since all the spaces are connected. □

Corollary 1.3 follows from Theorem 1.1 and the following lemma.

<u>Lemma 2.8.</u> If $q \geq n$, then $\pi_q(X;\mathbb{Z}/p)$ is isomorphic to $\pi_{q-n}(\mathrm{Map}_*(P^n(p),X))$ where $P^n(p) = S^{n-1} \cup_p e^n$.

<u>Proof:</u> $[X,Y]_*$ denotes homotopy classes of based maps $X \to Y$. Recall that $\pi_q(X,\mathbb{Z}/p) = [P^q(p),X]_*$. But $\pi_{q-n}(\mathrm{Map}_*(P^n(p),X)) = [S^{q-n},\mathrm{Map}_*(P^r(p),X)]_* = [S^{q-n} \wedge P^n(p),X]_* = [P^q(p),X]_*$. □

<u>Remark 2.9.</u> Notice that there is a fibration

$$S^5 \to SU(5)/Sp(2) \to S^9 .$$

Let $\partial\colon \Omega S^9 \to S^5$ be a choice of map induced by the above fibration. The map $f\colon W_2 \to \mathrm{Map}_*(\mathbb{R}P^2,\Omega S^5)$ defined above is homotopic to the composition

$$W_2 \xrightarrow{\sigma_2} \mathrm{Map}_*(\mathbb{R}P^2,\Omega^2 S^9) \xrightarrow{\mathrm{Map}_*(\mathbb{R}P^2,\Omega\partial)} \mathrm{Map}_*(\mathbb{R}P^2,\Omega S^5) .$$

The advantage of defining f as we did is that we easily obtain the product decomposition of $\mathrm{Map}_*(\mathbb{R}P^2,\Omega(SU(4)\langle 3\rangle))$ as well as the product decompositions for $\mathrm{Map}_*(\mathbb{R}P^2,\Omega^2 S^9)$ and $\mathrm{Map}_*(\mathbb{R}P^2,\Omega S^5)$.

§3. SPHERICAL CLASSES IN THE mod-p HOMOLOGY OF $\Omega^k(SU(n)\langle k\rangle)$ AND PROOFS OF STATEMENTS 1.8 AND 1.9

In Section 2 we defined $\alpha_n: S^{2n+1} \to (BSU(n))\langle k\rangle$ for $k \le 2n$ and it was shown that α_n was the generator of $\pi_{2n+1}(BSU(n))\langle k\rangle$. We show that if $j-1 \equiv n \bmod p$ and $1<j\le p$, then $(\Omega^{2j+1}\alpha_n)_*: H_*(\Omega^{2j+1}S^{2n+1};\mathbb{Z}/p) \to H_*(\Omega^{2j}(SU(n)\langle 2j\rangle);\mathbb{Z}/p)$ is non-zero in dimension $2n-2j$. Thus $(\Omega^{k+1}\alpha_n)_*: H_*(\Omega^{k+1}S^{2n+1};\mathbb{Z}/p) \to H_*(\Omega^k(SU(n)\langle k\rangle);\mathbb{Z}/p)$ is non-zero in dimension $2n-k$ if $2n > k \ge 2j$. Theorem 1.9, on the existence of spherical classes in $H_*(\Omega^k(SU(n)\langle k\rangle);\mathbb{Z}/p)$ follows from this statement.

By Bott periodicity, $\Omega^{2j-1}(SU\langle 2j-1\rangle)$ is homotopy equivalent to ΩSU, [1]. Thus $H_*(\Omega^{2j-1}(SU\langle 2j-1\rangle);\mathbb{Z}/p)$ is isomorphic to $P[w_2,w_4,w_5,\ldots]$ and the diagonal is given by $\Delta w_{2r} = \sum_{s=0}^{r} w_{2s}\otimes w_{2(r-s)}$ where $w_0 = 1$. Furthermore, the natural map of $\Omega^{2j-1}(SU(n+1)\langle 2j-1\rangle)$ to $\Omega^{2j-1}(SU\langle 2j-1\rangle)$ induces an isomorphism in dimensions $\le 2n-2j+2$. A well-known calculation [8] shows that a basis for the module of primitives in $H_*(\Omega^{2j-1}(SU\langle 2j-1\rangle);\mathbb{Z}/p)$ is given by the Newton polynomials, which we denote by $\bar{w}_{2r}$ for $r\ge 1$. $\bar{w}_{2r}$ is defined inductively by $\bar{w}_2 = w_2$ and $\bar{w}_{2r} = rw_{2r} - \sum_{s=1}^{r-1} w_{2s}\bar{w}_{2(r-s)}$.

Assume that $j-1 \equiv n \bmod p$ and $1<j\le p$. Then $PH_{2n-2j+2}(\Omega^{2j-1}(SU(n+1)\langle 2j-1\rangle);\mathbb{Z}/p)$ is generated by

$\bar{w}_{2(n-j+1)} = \sum_{s=1}^{n-j} w_{2s}\bar{w}_{2(n-j+1-s)}$. Let $p\colon SU(n+1)\langle 2j-1\rangle \to S^{2n+1}$ be the canonical fibration. Since $\Omega^{2j-1}p_*$ applied to w_{2s} or $\bar{w}_{2(n-j+1-s)}$ is zero for $1 \le s \le n-j$,
$\Omega^{2j-1}p_*(PH_{2n-2j+2}(\Omega^{2j-1}(SU(n+1)\langle 2j-1\rangle);\mathbb{Z}/p)) = 0$.

By applying the Serre spectral sequence to the morphism of fibrations

$$\begin{array}{ccccc}
\Omega^{2j+1}S^{2n+1} & \longrightarrow & * & \longrightarrow & \Omega^{2j}S^{2n+1} \\
\downarrow & & \downarrow & & \downarrow \\
\Omega^{2j}(SU(n)\langle 2j-1\rangle) & \longrightarrow & \Omega^{2j}(SU(n+1)\langle 2j-1\rangle) & \xrightarrow{\Omega^{2j}p} & \Omega^{2j}S^{2n+1}
\end{array}$$

there is an element z in $H_{2n-2j+1}(\Omega^{2j}(SU(n+1)\langle 2j-1\rangle);\mathbb{Z}/p)$ such that $\Omega^{2j}p_*(z) \neq 0$ if and only if $(\Omega^{2j+1}\alpha_n)_*(H_{2n-2j}(\Omega^{2j+1}S^{2n+1};\mathbb{Z}/p)) = 0$. But if such a z exists, then σz is primitive with non-zero image in $H_{2n-2j+2}(\Omega^{2j-1}S^{2n+1};\mathbb{Z}/p)$, where σ is the homology suspension. By the above paragraph, this cannot occur, so $\Omega^{2j+1}\alpha_*$ is non-zero on $H_{2n-2j}(-;\mathbb{Z}/p)$. Thus Theorem 1.9 follows.

The special case $j=2$ or 3 of the above is of interest in the homology computations that follow. This is recorded in the following corollary.

<u>Corollary 3.1.</u> The maps $\Omega^3\alpha_{pn}\colon \Omega^3S^{2pn+1} \to \Omega^2SU(pn)$ and $\Omega^4\alpha_{pn}\colon \Omega^4S^{2pn+1} \to \Omega^3(SU(pn)\langle 3\rangle)$ are non-zero on $H_{2pn-2}(-;\mathbb{Z}/p)$ and $H_{2pn-3}(-;\mathbb{Z}/p)$, respectively.

Recall that Theorem 1.8 specifies that certain elements in $H_*(\Omega^2 SU(n);\mathbb{Z}/p)$ and $H_*(\Omega^3(SU(pn)\langle 3\rangle);\mathbb{Z}/p)$ are in the image of Φ. This follows at once Corollary 3.1 and the descriptions of $H_*(\Omega^2 SU(pn);\mathbb{Z}/p)$ and $H_*(\Omega^3(SU(pn)\langle 3\rangle);\mathbb{Z}/p)$ given in Theorems 1.5 and 1.6.

§4. PREPARATORY RESULTS

The mod-p homology of $\Omega^k S^{2n+1}$ for $k < 2n+1$ plays a central role in computing the mod-p homology of the special unitary groups and their related homogeneous spaces. $H_*(\Omega^k S^{2n+1};\mathbb{Z}/p)$ is computed in [6].

<u>Definition 4.1.</u> A sequence $I = (\varepsilon_j, s_j, \ldots, \varepsilon_1, s_1)$ for $j \geq 0$ is said to be k-admissible if:

(i) $0 < s_i \leq s_{i-1} < k$.

(ii) If p=2, then $\varepsilon_i = 0$.

(iii) If p>2, then $\varepsilon_i = 0$ or 1 and $s_{i+1} \equiv s_i - \varepsilon_i \bmod 2$.

Q_I denotes the operation $\beta^{\varepsilon_j} Q_{s_j(p-1)} \cdots \beta^{\varepsilon_1} Q_{s_1(p-1)}$ where $I = (\varepsilon_j, s_j, \ldots, \varepsilon_1, s_1)$.

<u>Definition 4.2.</u> Let V be a graded set and let $k \geq 1$. Then (V,k) is the graded set

$$\left\{ Q_I v \;\middle|\; \begin{array}{l} v \in V,\ I = (\varepsilon_j, s_j, \ldots, \varepsilon_1, s_1) \text{ is k-admissible} \\ \text{and if } p>2, \text{ then } |v| \equiv s_1 \bmod 2 \end{array} \right\}.$$

<u>Definition 4.3.</u> Given a graded set V, we denote the symmetric algebra generated by the elements of (V,k) by S(V,k).

<u>Theorem 4.4.</u> Let $k < 2n+1$. There is a choice of primitive element $z_{n,k} \in H_{2n+1-k}(\Omega^k S^{2n+1};\mathbb{Z}/p)$ such that as a Hopf

algebra $H_*(\Omega^k S^{2n+1};\mathbb{Z}/p)$ is isomorphic to $S(\{z_{n,k}\},k)$. Furthermore, $z_{n,k}$ can be chosen so that $\sigma(z_{n,k}) = z_{n,k-1}$ where σ is the homology suspension.

Let $\Omega_0^n S^n$ be the zero component of $\Omega^n S^n$. To compute $H_*(\Omega^3(SU(n)\langle 3\rangle);\mathbb{Z}/p)$ we need to know the mod-p homology of $\Omega_0^3 S^3$. $H_*(\Omega_0^{2n+1} S^{2n+1};\mathbb{Z}/p)$ is given in [6].

Theorem 4.5. $H_*(\Omega_0^{2n+1}S^{2n+1};\mathbb{Z}/p) \subseteq H_*(\Omega^{2n+1}S^{2n+1};\mathbb{Z}/p)$ is isomorphic to the symmetric algebra generated by the set

$$\left\{ Q_I[1]*[-p^j] \;\middle|\; \begin{array}{l} I=(\varepsilon_j,s_j,\ldots,\varepsilon_1,s_1) \text{ is } 2n+1 \text{ admissible,} \\ j>0, \text{ and if } p>2, \text{ then } s_1 \equiv 0 \bmod 2 \end{array} \right\}$$

where [1] denotes the image of a generator of $\tilde{H}_0(S^0;\mathbb{Z}/p)$ under $S^0 \to \Omega^{2n+1}S^{2n+1}$ and $[r] = [1]^r$ for $r \in \mathbb{Z}$. The Pontryagin product is denoted by $*$.

Given a fibration $F \xrightarrow{i} X \xrightarrow{p} B$, we would like to express $H_*(\Omega^k X;\mathbb{Z}/p)$ in terms of $H_*(\Omega^k F;\mathbb{Z}/p)$, $H_*(\Omega^k B;\mathbb{Z}/p)$ and $\Omega^k f_*$ where $f\colon \Omega B \to F$ is a choice of map induced by the fibration p. We consider this when B is an odd sphere and there are certain restrictions on $\Omega^k f_*$.

If $F \xrightarrow{i} X \xrightarrow{p} S^{2n+1}$ is a fibration and $f\colon \Omega S^{2n+1} \to F$ is a choice of map induced by p, then the map $\Omega^k i_*$ factors as

$$\begin{array}{ccc} H_*(\Omega^k F;\mathbb{Z}/p) & \xrightarrow{\quad \Omega^k i_* \quad} & H_*(\Omega^k X;\mathbb{Z}/p) \\ & {\scriptstyle q} \searrow \qquad \nearrow {\scriptstyle g} & \\ & H_*(\Omega^k F;\mathbb{Z}/p) \,/\!/\, \mathrm{Im}\,\Omega^k f_* & \end{array}$$

where q is the quotient map and g is the unique map making the diagram commute. In the following proposition, the maps f and g are defined as above. If D is a subset of an algebra C, then $\langle D\rangle$ denotes the subalgebra of C generated by D.

<u>Proposition 4.6.</u> Let $F \xrightarrow{i} X \xrightarrow{p} S^{2n+1}$ be a fibration where F is k-connected and $2 \leq k \leq 2n-1$. Suppose that

(i) $\mathrm{Im}\,\Omega^k f_* \subseteq H_*(\Omega^k F;\mathbb{Z}/p)$ is a symmetric algebra.

(ii) If $p>0$ and $x \in \mathrm{Im}\,\Omega^k f_*$ is a primitive exterior generator, then $\beta Q_{(p-1)} x \neq 0$.

(iii) There is a set $V \subseteq \langle\sigma(\ker \Omega^k f_*)\rangle$ such that $\langle\sigma(\ker \Omega^k f_*)\rangle = S(V,j)$ for some $1 \leq j \leq k$.

Then the following properties hold.

(iv) $g\colon H_*(\Omega^k F;\mathbb{Z}/p) /\!/ \mathrm{Im}\,\Omega^k f_* \to H_*(\Omega^k X;\mathbb{Z}/p)$ is a monomorphism.

(v) There are choices of elements $x_v \in H_*(\Omega^k X;\mathbb{Z}/p)$ for $v \in V$ such that $\Omega^k p_*(x_v) = v$ and $S(\{x_v\}_{v\in V}, j)$ is a subalgebra of $H_*(\Omega^k X;\mathbb{Z}/p)$.

(vi) Identify $H_*(\Omega^k F;\mathbb{Z}/p)//\mathrm{Im}\,\Omega^k f_*$ with its image under g. Then $H_*(\Omega^k X;\mathbb{Z}/p)$ is isomorphic as an algebra to the tensor product of subalgebras,

$$H_*(\Omega^k F;\mathbb{Z}/p)//\mathrm{Im}\,\Omega^k f_* \otimes S(\{x_v\}_{v\in V}, j) \ .$$

Furthermore, if $V \subseteq \sigma(\mathrm{Im}\ \Omega^{k+1} p_*)$, then the x_v can be chosen to be primitive. Thus $S(\{x_v\}_{v\in V}, j)$ is a sub-Hopf algebra of $H_*(\Omega^k X;\mathbb{Z}/p)$ and the isomorphism in (vi) is a Hopf algebra isomorphism.

<u>Proof:</u> Consider the morphism of fibrations.

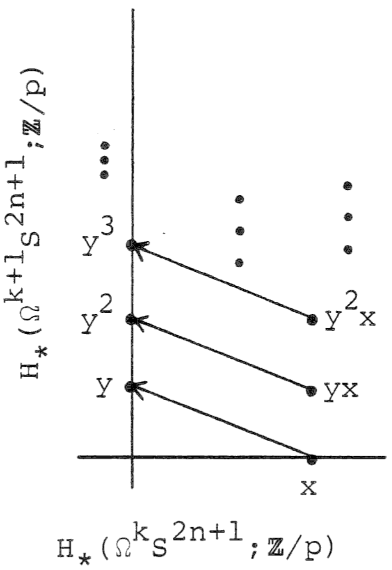
$$\begin{array}{ccccc}
\Omega^{k+1}S^{2n+1} & \longrightarrow & * & \longrightarrow & \Omega^k S^{2n+1} \\
\downarrow{\scriptstyle \Omega^k f} & & \downarrow & & \downarrow \\
\Omega^k F & \xrightarrow{\Omega^k i} & \Omega^k X & \xrightarrow{\Omega^k p} & \Omega^k S^{2n+1}
\end{array}$$

Since $k \le 2n-1$ and F is k-connected, all of the above spaces are 0-connected. We prove Proposition 4.6 by comparing the Serre spectral sequence for the top fibration to that of the bottom fibration.

The E^2 term for the Serre spectral sequence of the path space fibration of $\Omega^k S^{2n+1}$ is

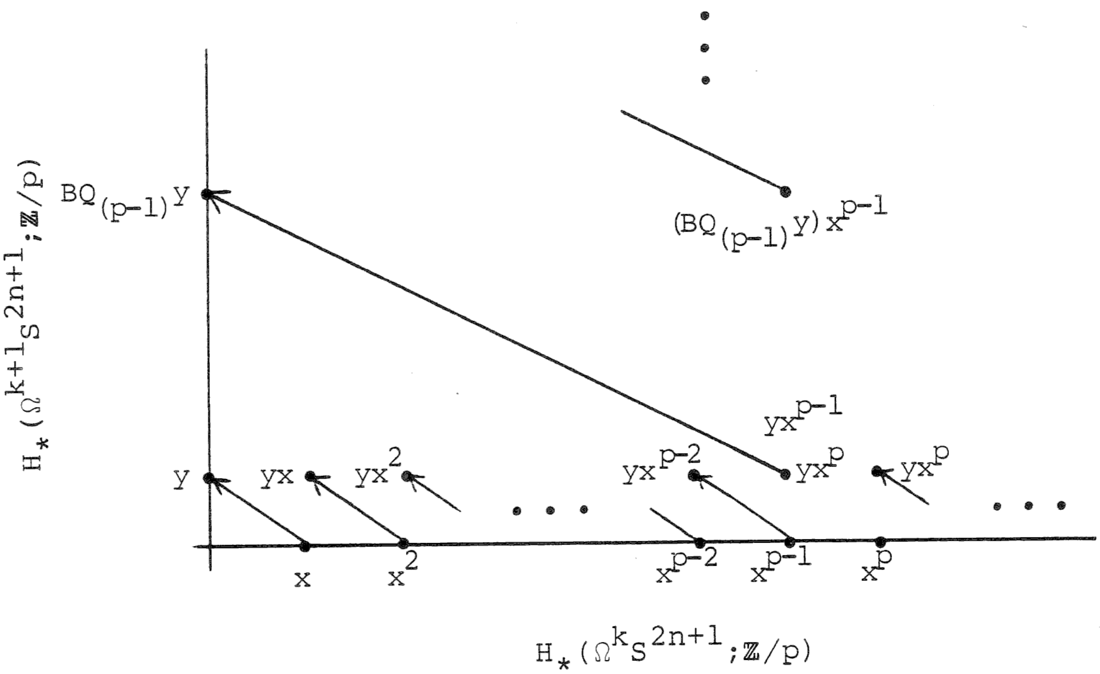
$$H_*(\Omega^{k+1}S^{2n+1};\mathbb{Z}/p) \otimes H_*(\Omega^k S^{2n+1};\mathbb{Z}/p) \ .$$

The differentials in this spectral sequence follow one of two patterns [6].



$H_*(\Omega^{k+1}S^{2n+1};\mathbb{Z}/p)$
y^3
y^2
y
y^2x
yx
x
$H_*(\Omega^k S^{2n+1};\mathbb{Z}/p)$

(a)



$H_*(\Omega^{k+1}S^{2n+1};\mathbb{Z}/p)$
$BQ_{(p-1)}y$
$(BQ_{(p-1)}y)x^{p-1}$
yx^{p-1}
y
yx
yx^2
yx^{p-2}
yx^p
yx^p
x
x^2
x^{p-2}
x^{p-1}
x^p
$H_*(\Omega^k S^{2n+1};\mathbb{Z}/p)$

(b)

The element x in $H_*(\Omega^k S^{2n+1};\mathbb{Z}/p)$ is primitive and y suspends to x. Type (a) differentials occur if p=2 or if p>2 and x is odd dimensional. Type (b) differentials occur if p>2 and x is even dimensional.

Conditions (i) and (ii) of Proposition 4.4 assure that in the Serre spectral sequence for the fibration

$$\Omega^k F \xrightarrow{\Omega^k i} \Omega^k X \xrightarrow{\Omega^k p} \Omega^k S^{2n+1}$$

that the differentials behave as above if $x \notin \sigma(\ker \Omega^k f_*)$, where y is replaced by $\Omega^k f_*(y)$. If $x \in \sigma(\ker \Omega^k f_*)$, then x is a permanent cycle. Thus the E^∞ term for this spectral sequence is

$$H_*(\Omega^k F;\mathbb{Z}/p) \,//\, \mathrm{Im}\, \Omega^k f_* \otimes \langle \sigma(\ker \Omega^k f_*) \rangle \;.$$

The E^∞ term gives the following information:

(i) $g\colon H_*(\Omega^k F;\mathbb{Z}/p)//\mathrm{Im}\,\Omega^k f_* \to H_*(\Omega^k X;\mathbb{Z}/p)$ is a monomorphism.

(ii) We may choose elements $x_v \in H_*(\Omega^k X;\mathbb{Z}/p)$ for $v \in V$ such that $\Omega^k p_*(x_v) = v$. Furthermore, if $V \subseteq \sigma(\mathrm{Im}\, \Omega^{k+1} p_*)$, then the x_v can be chosen to be in the image of the suspension, and hence are primitive. Since $H_*(\Omega^k X;\mathbb{Z}/p)$ is commutative and $\langle \sigma(\ker \Omega^k f_*) \rangle$ is the symmetric algebra $S(V,j)$, there are no algebra extension problems and $S(\{x_v\}_{v \in V}, j)$ is a subalgebra or sub-Hopf algebra of $H_*(\Omega^k X;\mathbb{Z}/p)$.

Thus $H_*(\Omega^k X;\mathbb{Z}/p)$ is isomorphic as an algebra, or Hopf algebra, to the tensor product

$$H_*(\Omega^k F;\mathbb{Z}/p) /\!/ \operatorname{Im} \Omega^k f_* \otimes S(\{x_v\}_{v\in V}, j) \ . \quad \square$$

<u>Remark 4.7.</u> $\langle \sigma(\ker \Omega^k f_*)\rangle$ is always a symmetric algebra. If V is assumed to be any set generating $\langle \sigma(\ker \Omega^k f_*)\rangle$ as a symmetric algebra and j is 1, then $\langle \sigma(\ker \Omega^k f_*)\rangle = S(V,j)$. Thus there is always a V and j as in Proposition 4.6(iii). However, the larger we are able to choose j, the more information we obtain concerning the Dyer-Lashof operations in $H_*(\Omega^k X;\mathbb{Z}/p)$.

<u>Remark 4.8.</u> We find it convenient to identify the element $x_v \in H_*(\Omega^k X;\mathbb{Z}/p)$ with the element $v \in H_*(\Omega^k S^{2n+1};\mathbb{Z}/p)$. This will simplify notation and make it easier to describe certain maps. It will be clear from the context whether we are considering v an element of $H_*(\Omega^k X;\mathbb{Z}/p)$ or an element of $H_*(\Omega^k S^{2n+1};\mathbb{Z}/p)$. It should be noted that the diagonal map and Steenrod operations on the element v in $H_*(\Omega^k X;\mathbb{Z}/p)$ are not in general the same as in $H_*(\Omega^k S^{2n+1};\mathbb{Z}/p)$.

The following two corollaries follow directly from Proposition 4.6. Corollary 4.9. describes the maps $\Omega^k i_*$ and $\Omega^k p_*$ where $F \xrightarrow{i} X \xrightarrow{p} S^{2n+1}$ and $f\colon \Omega S^{2n+1} \to F$ are as in Proposition 4.6.

<u>Corollary 4.9.</u> The map $\Omega^k i_*$ is given by the composition

$$\begin{array}{ccc} H_*(\Omega^k F;\mathbb{Z}/p) & \xrightarrow{\text{quotient}} & H_*(\Omega^k F;\mathbb{Z}/p) \,//\, \operatorname{Im}\Omega^k f_* \\ & & \downarrow \\ & & H_*(\Omega^k F;\mathbb{Z}/p) \,//\, \operatorname{Im}\Omega^k f_* \otimes S(V,j) \end{array}$$

where the vertical map sends x to $x\otimes 1$. The map $\Omega^k p_*$ is the quotient map composed with inclusion:

$$H_*(\Omega^k F;\mathbb{Z}/p) \,//\operatorname{Im}\Omega^k f_* \otimes S(V,j) \longrightarrow S(V,j) \hookrightarrow H_*(\Omega^k S^{2n+1};\mathbb{Z}/p).$$

In Corollary 4.10 we consider the morphism of fibrations

$$\begin{array}{ccccc} F_0 & \xrightarrow{i_0} & X_0 & \xrightarrow{p_0} & S^{2n+1} \\ \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle t} \\ F_1 & \xrightarrow{i_1} & X_1 & \xrightarrow{p_1} & S^{2n+1} \end{array}$$

where F_i is k-connected for some $2 \le k \le 2n-1$ and $f_i\colon \Omega S^{2n+1} \to F_i$ denotes a choice of map induced by p_i. We assume that $(\Omega^k f_i)_*$ satisfies conditions (i) and (ii) of Proposition 4.6, and that there exist sets $V_i \subseteq \langle\sigma(\ker(\Omega^k f_i)_*)\rangle$ such that $\langle\sigma(\ker(\Omega^k f_i)_*)\rangle = S(V_i,j_i)$ for some $1 \le j_i \le k$. Thus $H_*(\Omega^k X_i;\mathbb{Z}/p)$ is isomorphic to the tensor product

$$H_*(\Omega^k F_i;\mathbb{Z}/p) \,//\, \operatorname{Im}(\Omega^k f_i)_* \otimes S(V_i,j_i) .$$

To describe the map $\Omega^k s_*$, it suffices to give the action of $\Omega^k s_*$ on the elements in $H_*(\Omega^k F_0;\mathbb{Z}/p)/\!/\operatorname{Im}(\Omega^k f_0)_*$ and on the elements $v \in V_0$.

<u>Corollary 4.10.</u> The map $\Omega^k s_*$ is given by:

(i) Let $[y] \in H_*(\Omega^k F_i;\mathbb{Z}/p)/\!/\operatorname{Im}(\Omega^k f_i)_*$ denote the class of $y \in H_*(\Omega^k F_i;\mathbb{Z}/p)$. Then $\Omega^k s_*([y]) = [\Omega^k r_*(y)]$.

(ii) Let $v \in V_0$. Then $\Omega^k s_*(v) = \Omega^k t_*(v) + z_v$ where $z_v \in H_*(\Omega^k X_1;\mathbb{Z}/p)$ is in $\ker(\Omega^k p_1)_*$. Furthermore, if $\Omega^k t_*(V_0) \subseteq V_1$, then we may assume that the element $z_v = 0$.

<u>Remark 4.11.</u> To interpret the equation $\Omega^k s_*(v) = \Omega^k t_*(v) + z_v$, note that $\Omega^k t_*(v) \in S(V_{1,j}) \subseteq H_*(\Omega^k S^{2n+1};\mathbb{Z}/p)$. Thus following the conventions in Remark 4.8, we may consider $\Omega^k t_*(v)$ as an element in $H_*(\Omega^k X_1;\mathbb{Z}/p)$.

The following corollary is stated in anticipation of applying it along with the Bockstein Lemma in Section 10 to compute the higher Bocksteins in $H_*(\Omega^2(SU(n)/SU(m));\mathbb{Z}/p)$.

<u>Corollary 4.12.</u> Let $F \xrightarrow{i} X \xrightarrow{p} S^{2n+1}$ and $f\colon \Omega S^{2n+1} \to F$ be as in Proposition 4.6. If $x \in H_*(\Omega^k S^{2n+1};\mathbb{Z}/p)$ transgresses to $y \in H_*(\Omega^{k+1} S^{2n+1};\mathbb{Z}/p)$ in the Serre spectral sequence for the path space fibration of $\Omega^k S^{2n+1}$, then in the Serre spectral sequence for the fibration $\Omega^k p$, x transgresses to $\Omega^k f_*(y)$.

Proof: This follows from comparing the Serre spectral sequences for the following morphism of fibrations.

$$\begin{array}{ccccc} \Omega^{k+1}S^{2n+1} & \longrightarrow & * & \longrightarrow & \Omega^k S^{2n+1} \\ \downarrow & & \downarrow & & \downarrow \\ \Omega^k F & \longrightarrow & \Omega^k X & \longrightarrow & \Omega^k S^{2n+1} \end{array} . \square$$

§5. THE mod-p HOMOLOGY OF $\Omega^3(SU(n)\langle 3\rangle)$ AND PROOFS OF STATEMENTS 1.6 AND 2.5

To prove Theorem 1.6, which gives the algebra structure of $H_*(\Omega^3(SU(n)\langle 3\rangle);\mathbb{Z}/p)$, we induct on n using Proposition 4.6 applied to the fibration

$$SU(n)\langle 3\rangle \xrightarrow{i} SU(n+1)\langle 3\rangle \xrightarrow{p} S^{2n+1} .$$

Lemma 2.5 will follow general results obtained concerning the maps $\Omega^3 i_*$ and $\Omega^3 p_*$.

The following algebras are the building blocks for $H_*(\Omega^3(SU(n)\langle 3\rangle);\mathbb{Z}/p)$. Because of their importance, we give explicit descriptions of each.

<u>$S(U_i,3)$:</u> Let U_i be the graded set $\{u_i\}$ where $|u_i| = 2i-2$. By Theorem 4.4, $S(U_i,3)$ is isomorphic to $H_*(\Omega^3 S^{2i+1};\mathbb{Z}/p)$ where u_i is a generator of $H_{2i-2}(\Omega^3 S^{2i+1};\mathbb{Z}/p)$. When p=2,

$$S(U_i,3) = P[Q_1^a Q_2^b u_i \mid a,b\geq 0] .$$

When p>2,

$$S(U_i,3)$$
$$= P[Q^a_{2(p-1)} u_i \mid a\geq 0] \otimes E[Q^a_{(p-1)} \beta Q^b_{2(p-1)} u_i \mid a\geq 0,\ b>0]$$
$$\otimes P[\beta Q^a_{(p-1)} \beta Q^b_{2(p-1)} u_i \mid a,b>0] .$$

<u>$S(V_i,2)$:</u> The algebra $S(V_i,2)$ is a subalgebra of $S(U_i,3)$. Define $v_{i,a} = \beta Q^{a+1}_{2(p-1)} u_i$. When p=2, $\beta = Sq^1_*$. Thus $\beta Q_2^{a+1} u_i = Q_1 Q_2^a u_i$. Note that $|v_{i,a}| = 2p^{a+1}i - 3$.

Define $V_i = \{v_{i,a}\}_{a\geq 0}$. When p=2,

$$S(V_i,2) = P[Q_1^a v_{i,b} \mid a,b\geq 0] .$$

When p>2,

$$S(V_i,2)$$
$$= E[Q^a_{(p-1)} v_{i,b} \mid a,b\geq 0] \otimes P[\beta Q^a_{(p-1)} v_{i,b} \mid a>0,\ b\geq 0] .$$

<u>$S(U_i,3)/S(V_i,2)$:</u> For all p,

$$S(U_i,3)/S(V_i,2) = P[Q^a_{2(p-1)} u_i \mid a\geq 0] .$$

<u>S:</u> For all p,

$$S = P[Q^a_{2(p-1)} u \mid a\geq 0] \quad \text{where} \quad |u| = 2p-2 .$$

S arises as a quotient of $H_*(\Omega^3_0 S^3;\mathbb{Z}/p)$. Recall that $H_*(\Omega^3_0 S^3;\mathbb{Z}/p)$ was given in Theorem 4.5. Define a subalgebra $T \subseteq H_*(\Omega^3_0 S^3;\mathbb{Z}/p)$ as follows. When p=2,

$$T = P[Q_1^a Q_2^b [1]*[-2^{a+b}] \mid a>0,\ b\geq 0] .$$

When p>2,

$$T = E[Q^a_{(p-1)} \beta Q^b_{2(p-1)} [1]*[-p^{a+b}] \mid a\geq 0,\ b>0]$$
$$\otimes P[\beta Q^a_{(p-1)} \beta Q^b_{2(p-1)} [1]*[-p^{a+b}] \mid a,b\geq 0] .$$

Then $S = H_*(\Omega^3_0 S^3;\mathbb{Z}/p)/T$ where u is the class of $Q_{2(p-1)}[1]*[-p]$.

The next proposition, which implies both Theorem 1.6 and Lemma 2.5, is proved by induction. Much of the work in proving the inductive step is done in a series of technical lemmas concerning the Dyer-Lashof operations in $H_*(\Omega^3(SU(n)\langle 3\rangle);\mathbb{Z}/p)$. These lemmas are proved in Section 6. Because the proposition is used in the proofs of these lemmas, we index the proposition by n.

Proposition 5.1(n). Let $p<m\le n$. Then

(i) There are choices of elements u, u_i, and $v_{i,a}$ in $H_*(\Omega^3(SU(n)\langle 3\rangle);\mathbb{Z}/p)$ where

$$|u| = 2p-2, \qquad |u_i| = 2i-2, \qquad |v_{i,a}| = 2p^{a+1}i - 3,$$

and $H_*(\Omega^3(SU(n)\langle 3\rangle);\mathbb{Z}/p)$ is isomorphic to the following algebra:

$$S \otimes \left(\bigotimes_{\substack{1 < i \le \left\lceil \frac{n-1}{p} \right\rceil \\ i \not\equiv 0 \bmod p}} S(U_i,3)/S(V_i,2) \right) \otimes \left(\bigotimes_{\substack{\left\lceil \frac{n-1}{p} \right\rceil < i \le n-1 \\ i \not\equiv 0 \bmod p}} S(U_i,3) \right) \otimes \left(\bigotimes_{\substack{\left\lceil \frac{n-1}{p} \right\rceil < i \le n-1 \\ i \equiv 0 \bmod p}} S(V_i,2) \right)$$

(ii) Let $i\colon SU(m)\langle 3\rangle \to SU(m+1)\langle 3\rangle$ and $p\colon SU(m)\langle 3\rangle \to S^{2m-1}$ be the inclusion map and quotient map, respectively. Identify $H_*(\Omega^3(SU(m)\langle 3\rangle);\mathbb{Z}/p)$ with the isomorphism given above. Then the maps $\Omega^3 i_*$ and $\Omega^3 p_*$ are given as follows.

(a) Let $m \not\equiv 1 \bmod p$. The map $\Omega^3 i_*$ is the obvious inclusion map. The map $\Omega^3 p_*$ is the quotient map projecting onto the factor $S(U_{m-1}, 3)$.

(b) Let $m \equiv 1 \bmod p$. The map $\Omega^3 i_*$ is given by the quotient map $H_*(\Omega^3(SU(m)\langle 3\rangle);\mathbb{Z}/p) \to H_*(\Omega^3(SU(m)\langle 3\rangle);\mathbb{Z}/p) \,/\, S(V_{m/p}, 2)$ composed with the obvious inclusion map. The map $\Omega^3 p_*$ is the quotient map projecting onto the factor $S(V_{m-1}, 2)$.

Comparing the descriptions of $H_*(\Omega^3(SU(n)\langle 3\rangle);\mathbb{Z}/p)$ given in Theorem 1.6 and Proposition 5.1, we see that in Theorem 1.6 the generator in dimension $2p^{a+1}i - 3$ is $Q^a_{3(p-1)} v_{i,0}$, whereas in Proposition 5.1 the generator in dimension $2p^{a+1}i - 3$ is $v_{i,a}$.

<u>Lemma 5.2.</u> Let $p<n$. Then in $H_*(\Omega^3(SU(n)\langle 3\rangle);\mathbb{Z}/p)$, $Q^a_{3(p-1)} v_{i,0} = \varepsilon v_{i,a} +$ (decomposables) for some $\varepsilon \not\equiv 0 \bmod p$.

Theorem 1.6 follows from Proposition 5.1 and Lemma 5.2. The proof of Lemma 5.2 is given in Section 6. Lemma 2.5 follows directly from the second part of Proposition 5.1.

Proof of Proposition 5.1(p+1): Localized at the prime p, SU(p) is homotopy equivalent to $S^3 \times \cdots \times S^{2p-1}$ [11,15]. Thus $H_*(\Omega^3(SU(p)\langle 3\rangle);\mathbb{Z}/p)$ is isomorphic to $H_*(\Omega_0^3 S^3;\mathbb{Z}/p) \otimes H_*(\Omega^3 S^5;\mathbb{Z}/p) \otimes \cdots \otimes H_*(\Omega^3 S^{2p-1};\mathbb{Z}/p)$.

By Lemma 3.1, $(\Omega^4\alpha_p)_*$ is non-zero on $H_{2p-3}(-;\mathbb{Z}/p)$ where $\alpha_p\colon S^{2p+1} \to (BSU(p))\langle 4\rangle$ is defined as in Section 2. Let $z_{p,4}$ be a generator of $H_{2p-3}(\Omega^4 S^{2p+1};\mathbb{Z}/p)$. Because $H_{2p-3}(\Omega^3(SU(p)\langle 3\rangle);\mathbb{Z}/p)$ is generated by one element, $(\Omega^4\alpha_p)_*(z_{p,4}) = Q_1[1]*[-2]$ if p=2, and $(\Omega^4\alpha_p)_*(z_{p,4}) = \varepsilon(\beta Q_{2(p-1)}[1]*[-p])$ for some $\varepsilon \not\equiv 0 \bmod p$ if p>2. By the Cartan formula and the Adem relations,

$$
Q_1^aQ_2^bQ_3^c(Q_1[1]*[-2]) = \begin{cases} Q_1^{a+1}[1]*[-2^{a+1}] & \text{for } b,c=0 \\ Q_1^{a+1}Q_2^c[1]*[-2^{a+c+1}] \\ \quad + (Q_1^{a+c+1}[1]*[-2^{a+c+1}]) \\ \quad * (Q_1^c[1]*[-2^c]^{2^{a+1}} & \text{for } b=0,\ c>0 \\ (Q_1^{c+1}[1]*[-2^{c+1}])^4 & \text{for } b=1,\ a=0 \\ 0 & \text{otherwise} \end{cases}
$$

and

$$
\beta^{\varepsilon_1}Q^a_{(p-1)}\beta^{\varepsilon_2}Q^b_{2(p-1)}\beta^{\varepsilon_3}Q^c_{3(p-1)}(\beta Q_{2(p-1)}[1]*[-p])
$$

$$
= \begin{cases} \lambda\beta^{\varepsilon_1}Q^a_{(p-1)}\beta Q^{c+1}_{2(p-1)}[1]*[-p^{a+c+1}] & \text{for } \varepsilon_2,\varepsilon_3,b=0 \\ 0 & \text{otherwise} \end{cases}
$$

where $\lambda \not\equiv 0 \bmod p$. This determines the map $(\Omega^4\alpha_p)_*$. Thus the image of $(\Omega^4\alpha_p)_*$ is T and $\langle\sigma(\ker(\Omega^4\alpha_p)_*)\rangle$ is $S(V_p,2)$. Proposition 5.1(p+1) follows by applying Proposition 4.6 and Corollary 4.9 to the fibration

$$SU(p)\langle 3\rangle \xrightarrow{i} SU(p+1)\langle 3\rangle \xrightarrow{p} S^{2p+1} \ . \quad \square$$

The results obtained above concerning the image and kernel of $(\Omega^4\alpha_p)_*$ will be of use in Section 7. We record this in the following lemma.

Lemma 5.3. The image of $(\Omega^4\alpha_p)_*$ is T and $\langle\sigma(\ker(\Omega^4\alpha_p)_*)\rangle$ is $S(V_p,2)$.

Proof That Proposition 5.1(n) Implies Proposition 5.1(n+1): Let $n>p$. Assume Proposition 5.1(n). Proposition 5.1(n+1) follows by applying Proposition 4.6 and Corollary 4.9 along with the following lemma to the fibration

$$SU(n)\langle 3\rangle \xrightarrow{i} SU(n+1)\langle 3\rangle \xrightarrow{p} S^{2n+1} \ . \quad \square$$

Lemma 5.4. Let $n>p$.

(i) If $n \not\equiv 0 \bmod p$, then $\mathrm{Im}(\Omega^4\alpha_n)_* = 0$ and

$$\langle\sigma(\ker(\Omega^4\alpha_n)_*)\rangle = S(U_n,3) \ .$$

(ii) If $n \equiv 0 \bmod p$, then the ideal generated by $\mathrm{Im}(\Omega^4\alpha_n)_*$ is equal to the ideal generated by $S(V_{n/p},2)$ and $\langle\sigma(\ker(\Omega^4\alpha_n)_*)\rangle = S(V_n,2)$.

Lemma 5.4 is proved in Section 6. Thus assuming Lemma 5.4, we have shown Proposition 5.1(n) for all $n>p$.

§6. PROOFS OF LEMMAS 5.2 AND 5.4

Lemma 5.4, which describes the image and kernel of $(\Omega^4\alpha_n)_*$, was used in Section 5 to prove that Proposition 5.1(n) implies Proposition 5.1(n+1). Thus to prove Lemma 5.4, we assume Proposition 5.1(n).

Let $z_{n,4}$ be the generator of $H_{2n-3}(\Omega^4 S^{2n+1};\mathbb{Z}/p)$. To compute $(\Omega^4\alpha_n)_*$, it suffices to compute $(\Omega^4\alpha_n)_* z_{n,4}$ together with operations on this element.

To compute operations in $H_*(\Omega^3(SU(n)\langle 3\rangle;\mathbb{Z}/p)$, we need to have some hold on the module of primitives in $H_*(\Omega^3(SU(n)\langle 3\rangle);\mathbb{Z}/p)$.

Lemma 6.1. Let A be a connected Hopf algebra with both commutative multiplication and comultiplication over the field $\mathbb{Z}/p$. If $x \in A$ is primitive, then $x = y^{p^r}$ for some $r \geq 0$ where y projects to a non-zero element in the module of indecomposables.

Proof: Consider the exact sequence of Milnor-Moore [9],

$$0 \to P(\xi(A)) \to P(A) \to Q(A) \to Q((\xi(A^*))^*) \to 0$$

where ξ is the p^{th}-power map. If x is primitive and projects to zero in Q(A), then $x \in \xi(A)$. Now apply the exact sequence of Milnor-Moore to the Hopf algebra $\xi(A)$. Since x is not in $\xi^r(A)$ for all r, x must project to a non-zero element in $Q(\xi^r(A))$ for some $r \geq 0$. Thus $x = y^{p^r}$ and y projects to a non-zero element in $Q(A)$. □

We list the generators of $H_*(\Omega^3(SU(n)\langle 3\rangle);\mathbb{Z}/p)$ given by Proposition 5.1(n) and give dimensions in which these elements lie. The cases p=2 and p 2 are considered separately.

(i) Let p=2.

$$|Q_2^a u| = 2^{a+2}-2 \quad \text{for } a \geq 0$$

$$|Q_2^a u_i| = 2^{a+1}i-2 \text{ for } a \geq 0,\ 2 \leq i \leq n-1,\ i \not\equiv 0 \bmod 2$$

$$|Q_1^{a+1} Q_2^b u_i|$$
$$= 2^a(2^{b+2}i-2)-1 \text{ for } a,b \geq 0,\ \left[\frac{n-1}{2}\right] < i \leq n-1,\ i \not\equiv 0 \bmod 2$$

$$|Q_1^a v_{i,b}|$$
$$= 2^a(2^{b+2}i-2)-1 \text{ for } a,b \geq 0,\ \left[\frac{n-1}{2}\right] < i \leq n-1,\ i \equiv 0 \bmod 2$$

(ii) Let p>2.

$$|Q_{2(p-1)}^a u| = 2p^{a+1}-2 \quad \text{for } a \geq 0$$

$$|Q_{2(p-1)}^a u_i| = 2p^a i-2 \quad \text{for } a \geq 0,\ 2 \leq i \leq n-1,\ i \not\equiv 0 \bmod p$$

$$|Q_{(p-1)}^a \beta Q_{2(p-1)}^{b+1} u_i| = p^a(2p^{b+1}i-2)-1 \quad \text{and}$$

$$|\beta Q_{(p-1)}^{a+1} \beta Q_{2(p-1)}^{b+1} u_i| = p^{a+1}(2p^{b+1}i-2)-2$$

$$\text{for } a,b \geq 0,\ \left[\frac{n-1}{p}\right] < i \leq n-1,\ i \not\equiv 0 \bmod p$$

$$|Q^a_{(p-1)} v_{i,b}| = p^a(2p^{b+1}i-2)-1 \quad \text{and}$$

$$|\beta Q^{a+1}_{(p-1)} v_{i,b}| = p^{a+1}(2p^{b+1}i-2)-2$$

$$\text{for } a,b \geq 0, \left[\frac{n-1}{p}\right] < i \leq n-1,\ i \equiv 0 \bmod p.$$

By Lemma 6.1 and inspection of the dimensions in which the generators of $H_*(\Omega^3(SU(n)\langle 3\rangle);\mathbb{Z}/p)$ lie, we obtain the following lemma.

Lemma 6.2. (i) The module of indecomposables of $H_*(\Omega^3(SU(n)\langle 3\rangle);\mathbb{Z}/p)$ has at most one generator in any dimension.

(ii) Let $n=pk$ where $k>1$. Then any primitive in $H_{2p^{a+1}k-3}(\Omega^3(SU(n)\langle 3\rangle);\mathbb{Z}/p)$ is of the form $\lambda(v_{i,a}+t)$ where the image of t in $QH_*(\Omega^3(SU(n)\langle 3\rangle);\mathbb{Z}/p)$ is zero.

(iii) Let $n=pk$ where $k>1$ and $p>2$. Then $PH_{2p^{a+2}k-4}(\Omega^3(SU(n)\langle 3\rangle);\mathbb{Z}/p) = 0$.

(iv) Let $n=2k$ where $k>1$. Then any element in $PH_{2^{a+3}k-4}(\Omega^3(SU(n)\langle 3\rangle);\mathbb{Z}/2)$ is of the form:

$$\lambda(Q_2^{c+a}u + t)^2 \qquad \text{if } k = 2^c,\ c>1$$

$$\lambda(Q_2^{c+a+1}u_i + t)^2 \quad \text{if } k = 2^c i,\ i \not\equiv 0 \bmod 2,\ i>1$$

where t projects to zero in $QH_*(\Omega^3(SU(n)\langle 3\rangle);\mathbb{Z}/2)$.

Lemma 6.3. Let $n=2k$ where $k>1$. Consider the elements in $H_*(\Omega^3(SU(n)\langle 3\rangle);\mathbb{Z}/2)$ of the form

$$\lambda(Q_2^{c+a}u+t)^2 \qquad \text{for } k=2^c,\ c>1$$

$$\lambda(Q_2^{c+a+1}u_i+t)^2 \qquad \text{for } k=2^c i,\ i\not\equiv 0 \bmod 2,\ i>1$$

where t projects to zero in $QH_*(\Omega^3(SU(n)<3>);\mathbb{Z}/2)$. Then Sq_*^4 applied to either of the above elements is zero if and only if $\lambda\equiv 0 \bmod 2$.

Proof: Apply Sq_*^4 to the above elements to obtain the following:

$$\lambda((Q_2^{c+a-1}u)^2+Sq_*^2t)^2 = Sq_*^4(\lambda(Q^{c+a}u+t)^2)$$

$$\lambda((Q_2^{c+a}u_i)^2+Sq_*^2t)^2 = Sq_*^4(\lambda(Q^{c+a+1}u_i+t)^2) .$$

Since t projects to zero in the module of indecomposables, $t=\sum_j r_js_j$ where $|r_j|,|s_j|>0$. Thus $Sq_*^2t = \sum_j Sq_*^2r_js_j^2 + Sq_*^1r_j\, Sq_*^1s_j + r_j^2\, Sq_*^2s_j$. Hence if Sq_*^2t is a square, then for some j, $r_j = s_j$ and r_j projects non-trivially in the module of indecomposables. Thus $|r_j| = 2^{a+1}k-1$. By inspection of the dimensions in which the generators of $H_*(\Omega^3(SU(n)<3>);\mathbb{Z}/2)$ lie, there can be no such r_j. Thus the above elements are zero if and only if $\lambda\equiv 0 \bmod 2$. □

Lemma 6.4. Let n=pk where k>1. Then $(\Omega^4\alpha_n)_*\, z_{n,4}=\varepsilon v_{k,0}$ for some $\varepsilon\not\equiv 0 \bmod p$.

Proof: By Lemma 3.1, $(\Omega^4\alpha_n)_*\, z_{n,4}\neq 0$. Since $v_{k,0}$ generates $H_{2n-3}(\Omega^3(SU(n)<3>);\mathbb{Z}/p)$, Lemma 6.4 follows. □

Lemma 6.5. Let $n=pk$ where $k>1$. Then in $H_*(\Omega^3(SU(n)\langle 3\rangle);\mathbb{Z}/p)$, $Q^a_{3(p-1)}v_{k,0} = \varepsilon_a v_{k,a} + t_a$ where $\varepsilon_a \not\equiv 0$ mod p and t_a projects to zero in $QH_*(\Omega^3(SU(n)\langle 3\rangle);\mathbb{Z}/p)$.

Proof: Proceed by induction on a. If $a=0$, then Lemma 6.4 is clearly true. Let $a>0$. Assume $Q^{a-1}_{3(p-1)}v_{k,0} = \varepsilon_{a-1}v_{k,a-1} + t_{a-1}$ where $\varepsilon_{a-1} \not\equiv 0$ mod p and t_{a-1} projects to zero in the module of indecomposables. By the Nishida relations applied in $H_*(\Omega^4S^{2n+1};\mathbb{Z}/p)$, the following formulas hold: If $p=2$, then $Sq^2_*Q^a_3z_{n,4} = Q_1Q^{a-1}_3z_{n,4}$ and if $p>2$, then $P^1_*Q^a_{3(p-1)}z_{n,4} = \lambda Q_{(p-1)}Q^{a-1}_{3(p-1)}z_{n,4}$ for some $\lambda \not\equiv 0$ mod p. By applying $(\Omega^4\alpha_n)_*$ to the above equations together with the induction hypothesis and Lemma 6.4, the following formulas hold: If $p=2$, then $Sq^2_*Q^a_3v_{k,0} = Q_1v_{k,a-1} + Q_1t_{a-1}$ and if $p>2$, then $P^1Q^a_{3(p-1)}v_{k,0} = \lambda\varepsilon_{a-1}Q_1v_{k,a-1} + Q_1t_{a-1}$. Thus for all p, $Q^a_{3(p-1)}v_{k,0} \neq 0$. Also $Q^a_{3(p-1)}v_{k,0}$ is primitive because it is the image of a primitive. Thus by Lemma 6.2(ii), $Q^a_{3(p-1)}v_{k,0} = \varepsilon_a v_{k,a} + t_a$ for some $\varepsilon_a \not\equiv 0$ mod p and where the image of t_a in $QH_*(\Omega^3(SU(n)\langle 3\rangle);\mathbb{Z}/p)$ is zero. □

Lemma 6.6. Let $n=pk$ where $k>1$. Then in $H_*(\Omega^3(SU(n)\langle 3\rangle);\mathbb{Z}/p)$, the following formulas hold:

(i) Let $p=2$. Then $Q_2Q^a_3v_{k,0} = 0$.

(ii) Let $p>2$. Then $\beta Q^{a+1}_{3(p-1)}v_{k,0} = 0$.

Proof: Let p=2. By Lemma 6.4, $(\Omega^4\alpha_n)_*(Q_2Q_3^a z_{n,4}) = Q_2Q_3^a v_{k,0}$. Thus $Q_2Q_3^a v_{k,0}$ is primitive. By Lemma 6.2(iv),

$$Q_2Q_3^a v_{k,0} = \begin{cases} \lambda(Q_2^{c+a}u + t)^2 & \text{if } k=2^c,\ c>1 \\ \lambda(Q_2^{c+a+1}u_j + t)^2 & \text{if } k=2^c j,\ j \not\equiv 0 \bmod 2,\ j>1 \end{cases}$$

where t projects to zero in $QH_*(\Omega^3(SU(n)\langle 3\rangle);\mathbb{Z}/2)$. By Lemma 6.3, SQ_*^4 applied to the right-hand side of the above equation is zero if and only if $\lambda \equiv 0 \bmod 2$.

By the Nishida relations, in $H_*(\Omega^4 S^{2n+1};\mathbb{Z}/2)$

$$Sq_*^4(Q_2Q_3^a z_{n,4}) = \begin{cases} 0 & \text{if } a=0 \\ Q_2Q_3^{a-1}z_{n,4} & \text{if } a>0 \end{cases}$$

By applying $(\Omega^4\alpha_n)_*$, we see that $Sq_*^4(Q_2v_{k,0}) = 0$ and if $Sq_*^4(Q_2Q_3^a v_{k,0}) = 0$, then $Sq_*^4(Q_2Q_3^{a+1}v_{k,0}) = 0$. Thus $Q_2Q_3^a v_{k,0} = 0$ for all $a>0$. This proves Lemma 6.6(i).

Since $\beta Q_{3(p-1)}^{a+1}v_{k,0}$ is primitive and in dimension $2p^{a+2}k-4$, Lemma 6.6(ii) follows from Lemma 6.2(iii). □

Proof of Lemma 5.4: Let $n \not\equiv 0 \bmod p$. By Proposition 5.1(n), $H_{2n-3}(\Omega^3(SU(n)\langle 3\rangle);\mathbb{Z}/p) = 0$. Thus $(\Omega^4\alpha_n)_* z_{n,4} = 0$, which implies $(\Omega^4\alpha_n)_* = 0$. Lemma 5.4(i) follows from this statement.

Let n=kp for k>1. By Lemma 6.4, $(\Omega^4\alpha_n)_* z_{n,4} = \varepsilon v_{k,0}$. Thus by Lemmas 6.5 and 6.6, the image of $(\Omega^4\alpha_n)_*$ is generated by elements of the form $Q^a_{(p-1)}(v_{k,b}+t_b)$ and $\beta Q^{a+1}_{(p-1)}(v_{k,b}+t_b)$ where t_b projects to zero in the module of indecomposables. By Proposition 5.1(n), the image of $H_*(\Omega^3(SU(k+1)\langle 3\rangle;\mathbb{Z}/p)$ in $H_*(\Omega^3(SU(n)\langle 3\rangle);\mathbb{Z}/p)$ is

$$S \otimes \left[\bigotimes_{\substack{1 < i \le k-1 \\ i \not\equiv 0 \bmod p}} S(U_i,3)/S(V_i,2) \right] \otimes \begin{cases} S(U_k,3) & \text{if } k \not\equiv 0 \bmod p \\ S(V_k,2) & \text{if } k \equiv 0 \bmod p \end{cases} .$$

t_b is in the image of $H_*(\Omega^3(SU(k+1)\langle 3\rangle);\mathbb{Z}/p)$ and is odd dimensional. Thus it is in the ideal generated by $S(V_k,2)$. This implies that the ideal generated by $\mathrm{Im}(\Omega^4\alpha_n)_*$ is equal to the ideal generated by $S(V_k,2)$. That $\langle\sigma(\ker(\Omega^4\alpha_n)_*)\rangle =$ $S(V_n,2)$ follows directly from Lemmas 6.4, 6.5 and 6.6. □

<u>Proof of Lemma 5.2:</u> We show that in $H_*(\Omega^3(SU(n)\langle 3\rangle);\mathbb{Z}/p)$, $Q^a_{3(p-1)}v_{i,0} = \varepsilon v_{i,a} +$ (decomposables) where $\varepsilon \not\equiv 0 \bmod p$. Consider the image of $v_{i,0}$ in $H_*(\Omega^3(SU(pi));\mathbb{Z}/p)$. By Lemma 6.5, $Q^a_{3(p-1)}v_{i,0} = \varepsilon v_{i,a} +$ (decomposables) where $\varepsilon \not\equiv 0 \bmod p$. Thus by Lemma 6.2(i) and the above, $Q^a_{3(p-1)}v_{i,0} =$ $\varepsilon v_{i,a} +$ (decomposables) in $H_*(\Omega^3(SU(n)\langle 3\rangle);\mathbb{Z}/p)$.

§7. THE mod-p HOMOLOGY OF $\Omega^3(SU(n)/SU(m))$ AND THE PROOF OF THEOREM 1.11

The technique used to prove Theorem 1.11, which gives the algebra structure of $H_*(\Omega^3(SU(n)/SU(m));\mathbb{Z}/p)$, is similar to that used in Section 5. The following proposition implies Theorem 1.11.

Proposition 7.1. Let $2 \le m < n$. Then

(i) There are choices of elements u_i and $v_{i,a}$ in $H_*(\Omega^3(SU(n)/SU(m));\mathbb{Z}/p)$ where

$$|u_i| = 2i-2 \qquad |v_{i,a}| = 2p^{a+1}i-3$$

and $H_*(\Omega^3(SU(n)/SU(m));\mathbb{Z}/p)$ is isomorphic to the following algebra:

$$\left(\bigotimes_{i\in A_p(n,m)} S(U_i,3)/S(V_i,2)\right) \otimes \left(\bigotimes_{i\in B_p(n,m)} S(U_i,3)\right) \otimes \left(\bigotimes_{i\in C_p(n,m)} S(V_i,2)\right)$$

where the indexing sets $A_p(n,m)$, $B_p(n,m)$ and $C_p(n,m)$ are as defined in Section 1.

(ii) Let $j_{n,m}: SU(n)\langle 3\rangle \to SU(n)/SU(m)$ be the natural quotient map. If $n>p$, then $(\Omega^3 j_{n,m})_*$ is given by:

(a) $(\Omega^3 j_{n,m})_* u = 0$

(b) $(\Omega^3 j_{n,m})_* u_i = \begin{cases} u_i & i \ge m \\ 0 & i < m \end{cases}$

(c) $$(\Omega^3 j_{n,m})_* v_{i,a} = \begin{cases} v_{i,a} & i > p(m-1) \\ v_{i,a} + (\text{decomposables}) & m \le i \le p(m-1) \\ 0 & i < m \end{cases}$$

(iii) Let $i: SU(n-1)/SU(m) \to SU(n)/SU(m)$ be the natural inclusion. Identify $H_*(\Omega^3(SU(n)/SU(m)); \mathbb{Z}/p)$ with the isomorphism given above. Then the map $\Omega^3 i_*$ is given by the composition of the natural quotient and inclusion maps.

Proposition 7.1 is proved by induction. The following lemma provides most of the information needed to carry through the induction step. We assume Proposition 7.1 for $2 \le m < n$, where $n \ge p$. Let $\gamma_{n,m}: \Omega S^{2n+1} \to SU(n)/SU(m)$ be a choice of map induced by the fibration $p: SU(n+1)/SU(m) \to S^{2n+1}$.

<u>Lemma 7.2.</u> Let $2 \le m < n$ where $n \ge p$.

(i) If $n \not\equiv 0 \bmod p$ or $m > [n/p]$, then $\mathrm{Im}(\Omega^3 \gamma_{n,m})_* = 0$ and $\langle \sigma(\ker(\Omega^3 \gamma_{n,m})_*) \rangle = S(U_n, 3)$.

(ii) If $n \equiv 0 \bmod p$ and $m \le [n/p]$, then the ideal generated by $\mathrm{Im}(\Omega^3 \gamma_{n,m})_*$ is equal to the ideal generated by $S(V_{n/p}, 2)$ and $\langle \sigma(\ker(\Omega^3 \gamma_{n,m})_*) \rangle = S(V_n, 2)$.

<u>Proof:</u> The map $\gamma_{n,m}$ factors as:

$$\begin{array}{ccc} \Omega S^{2n+1} & \xrightarrow{\gamma_{n,m}} & SU(n)/SU(m) \\ & {\scriptstyle \Omega\alpha_n} \searrow \quad \nearrow {\scriptstyle j_{n,m}} & \\ & SU(n)\langle 3\rangle & \end{array}$$

Proposition 7.1(ii) gives $(\Omega^3 j_{n,m})_*$ except when $n=p$. Localized at the prime p, $\Omega^3(SU(p)\langle 3\rangle)$ is homotopy equivalent to $\Omega_0^3 S^3 \times \Omega^3 S^5 \times \cdots \times \Omega^3 S^{2p-1}$ and $\Omega^3(SU(p)/SU(m))$ is homotopy equivalent to $\Omega^3 S^{2m+1} \times \cdots \times \Omega^3 S^{2p-1}$ and $\Omega^3 j_{n,m}$ is projection [11,15]. Thus $(\Omega^3 j_{n,m})_*$ is the obvious quotient map,

$$H_*(\Omega_0^3 S^3;\mathbb{Z}/p) \otimes H_*(\Omega^3 S^5;\mathbb{Z}/p) \otimes \cdots \otimes H_*(\Omega^3 S^{2p-1};\mathbb{Z}/p)$$
$$\to H_*(\Omega^3 S^{2m+1};\mathbb{Z}/p) \otimes \cdots \otimes H_*(\Omega^3 S^{2p-1};\mathbb{Z}/p) .$$

Information concerning $(\Omega^4\alpha_n)_*$ is given by Lemma 5.3 when $n=p$, and by Lemma 5.4 when $n>p$.

Let $n=p$. Then $m > [n/p]$. By Lemma 5.3, $\mathrm{Im}(\Omega^4\alpha_n)_* = T \subseteq H_*(\Omega_0^3 S^3);\mathbb{Z}/p)$. By the above remarks, $(\Omega^3 j_{p,m})_*$ is zero on this factor. Thus $(\Omega^3\gamma_{p,m})_* = 0$.

Let $n>p$. If $n \not\equiv 0 \bmod p$, then by Lemma 5.4, $(\Omega^4\alpha_n)_* = 0$. Thus $(\Omega^3\gamma_{n,m})_* = 0$. If $n \equiv 0 \bmod p$ and $m > [n/p]$, then by Proposition 7.1, $(\Omega^3 j_{n,m})_*(S(V_{n/p},2)) = 0$. By Lemma 5.4, the ideal generated by $\mathrm{Im}(\Omega^4\alpha_n)_*$ is equal to the ideal generated by $S(V_{n/p},2)$. Thus $(\Omega^3 j_{n,m})_* = 0$.

Hence we have shown if $n \equiv 0 \bmod p$ or $m > [n/p]$, then $(\Omega^3 j_{n,m})_* = 0$. Lemma 7.2(i) follows from this statement.

Let $n>p$ where $n=kp$ and $m\le k$. By Proposition 7.1(iii), $(\Omega^3 j_{n,m})_* v_{k,a}$ is contained in the image of $H_*(\Omega^3(SU(k+1)/SU(m));\mathbb{Z}/p)$ in $H_*(\Omega^3(SU(n)/SU(m));\mathbb{Z}/p)$, which is

$$\left(\bigotimes_{i\in A_p(n,m)} S(U_i,3)/S(V_i,2)\right)\otimes\begin{cases} S(U_k,3) & k\not\equiv 0 \bmod p \\ S(V_k,3) & k\equiv 0 \bmod p\,. \end{cases}$$

Since $(\Omega^3 j_{n,m})_* v_{k,a}$ is odd dimensional, it must be in the ideal generated by $S(V_k,2)$. By Proposition 7.1(ii), $(\Omega^3 j_{n,m})(v_{k,a}) = v_{k,a}$ + (decomposables). Thus the ideal generated by $(\Omega^3 j_{n,m})_*(S(V_{k,2}))$ is equal to the ideal generated by $S(V_k,2)$. By Lemma 5.4, the ideal generated by $\mathrm{Im}(\Omega^4\alpha_n)_*$ is equal to the ideal generated by $S(V_k,2)$. Thus the ideal generated by $\mathrm{Im}(\Omega^3\gamma_{n,m})_*$ is equal to the ideal generated by $S(V_k,2)$. Also, by Proposition 7.1(ii), we see that $(\Omega^3 j_{n,m})_*$ is a monomorphism on $\mathrm{Im}(\Omega^4\alpha_n)_*$. Thus $\langle\sigma(\ker(\Omega^3\gamma_{n,m})_*)\rangle = \langle\sigma(\ker(\Omega^4\alpha_n)_*)\rangle = S(V_n,2)$. □

Proof of Proposition 7.1: Proceed by induction on n. If $2\le m<n\le p$, then localized at p, $\Omega^3(SU(n)/SU(m))$ is homotopy equivalent to $\Omega^3 S^{2m+1}\times\cdots\times\Omega^3 S^{2n-1}$ and $\Omega^3 i\colon \Omega^3(SU(n-1)/SU(m))\to\Omega^3(SU(n)/SU(m))$ is homotopy equivalent to the inclusion of $\Omega^3 S^{2m+1}\times\cdots\times\Omega^3 S^{2n-3}$ into $\Omega^3 S^{2m+1}\times\cdots\times\Omega^3 S^{2n-1}$. Proposition 7.1 for $n\le p$ now follows by applying $H_*(-;\mathbb{Z}/p)$.

Let $n \geq p$. Assume Proposition 7.1 for n. Proposition 7.1(i) and (iii) for n+1 follow by applying Proposition 4.6 and Corollary 4.9 along with Lemma 7.2 to the fibration

$$SU(n)/SU(m) \to SU(n+1)/SU(m) \to S^{2n+1} .$$

To determine the map $(\Omega^3 j_{n+1,m})_*$ apply Corollary 4.10 to the morphism of fibrations

$$\begin{array}{ccccc} SU(n)\langle 3\rangle & \longrightarrow & SU(n+1)\langle 3\rangle & \longrightarrow & S^{2n+1} \\ \downarrow{\scriptstyle j_{n,m}} & & \downarrow{\scriptstyle j_{n+1,m}} & & \downarrow{\scriptstyle id} \\ SU(n)/SU(m) & \longrightarrow & SU(n+1)/SU(m) & \xrightarrow{p} & S^{2n+1} \end{array} .$$

This gives $(\Omega^3 j_{n+1,m})_*$ as in Proposition 7.1(iii) on the image of $H_*(\Omega^3(SU(n)\langle 3\rangle);\mathbb{Z}/p)$ in $H_*(\Omega^3(SU(n+1)\langle 3\rangle);\mathbb{Z}/p)$. However, on the factor $\langle \sigma(\ker(\Omega^4 \alpha_n)_*)\rangle$ of $H_*(\Omega^3(SU(n+1)\langle 3\rangle;\mathbb{Z}/p)$, there is indeterminancy. If $m \leq [n/p]$ or $n \not\equiv 0 \bmod p$, then this indeterminancy is zero by Corollary 4.10(ii) and the fact that $\langle \sigma(\ker(\Omega^4\alpha_n)_*)\rangle = \langle \sigma(\ker(\Omega^3\gamma_{n,m})_*)\rangle$. If $m > [n/p]$ and $n \equiv 0 \bmod p$, the indeterminancy is present and is given by $\ker \Omega^3 p_*$. Since $QH_{2p^{a+1}n-3}(\Omega^3(SU(n+1)/SU(m));\mathbb{Z}/p)$ is generated by the image of $v_{n,a}$ and $\Omega^3 p_* v_{n,a} \neq 0$, the indeterminancy consists of decomposable elements. Thus Proposition 7.1(iii) follows. □

§8. THE mod-p HOMOLOGY OF $\Omega^2(SU(n)/SU(m))$ AND PROOFS OF STATEMENTS 1.5, 1.10, 1.12, 1.13 AND 1.15

In this section we prove Theorem 1.4 and Theorem 1.10 which give $H_*(\Omega^2(SU(n)/SU(m));\mathbb{Z}/p)$ as a Hopf algebra, Proposition 1.14 and 1.15 concerning the Dyer-Lashof and Browder operations in $H_*(\Omega^2(SU(n)/SU(m));\mathbb{Z}/p)$ and Proposition 1.12 which describes the map $\Omega^2 j_*$ where $j\colon SU(n) \to SU(n)/SU(m)$ is the natural quotient map.

The following Hopf algebras are the building blocks for $H_*(\Omega^2(SU(n)/SU(m));\mathbb{Z}/p)$. Because of their importance, we give explicit descriptions of each.

<u>$S(X_i,2)$:</u> Let X_i be the graded set $\{x_i\}$ where $|x_i| = 2i-1$. By Theorem 4.4, $S(X_i,2)$ is isomorphic to $H_*(\Omega^2 S^{2i+1};\mathbb{Z}/p)$ where x_i is the generator of $H_{2i-1}(\Omega^2 S^{2i+1};\mathbb{Z}/p)$. When p=2,

$$S(X_i,2) = P[Q_1^a x_i \mid a \geq 0] .$$

When p>2,

$$S(X_i,2) = E[Q^a_{(p-1)} x_i \mid a \geq 0] \otimes P[\beta Q^a_{(p-1)} x_i \mid a > 0] .$$

<u>$S(Y_i,1)$:</u> The Hopf-algebra $S(Y_i,1)$ is a sub-Hopf algebra of $S(X_i,2)$. Define $y_{i,a} = \beta Q^{a+1}_{(p-1)} x_i$. When p=2, $\beta = Sq^1_*$. Thus $\beta Q_1^{a+1} x_i = (Q_1^a x_i)^2$. Note that $|y_{i,a}| = 2p^{a+1}i-2$. Define $Y_i = \{y_{i,a}\}_{a\geq 0}$. For all p,

$$S(Y_i,1) = P[y_{i,a} \mid a \geq 0] .$$

$S(X_i,2)/S(Y_i,1)$: For all p,

$$S(X_i,2)/S(Y_i,1) = E[Q^a_{(p-1)} x_i \mid a\geq 0] .$$

The following proposition directly implies Theorem 1.10.

Proposition 8.1. Let $1\leq m<n$. Then

(i) There are choices of elements x_i and $y_{i,a}$ in $H_*(\Omega^2(SU(n)/SU(m));\mathbb{Z}/p)$ where

$$|x_i| = 2i-1 , \qquad |y_{i,a}| = 2p^{a+1}i-2$$

and $H_*(\Omega^2(SU(n)/SU(m));\mathbb{Z}/p)$ is isomorphic to the following Hopf algebra:

$$\left(\bigotimes_{i\in A_p(n,m)} S(X_i,2)/S(Y_i,1)\right) \otimes \left(\bigotimes_{i\in B_p(n,m)} S(X_i,2)\right) \otimes \left(\bigotimes_{i\in C_p(n,m)} S(Y_i,1)\right)$$

where the indexing sets $A_p(n,m)$, $B_p(n,m)$ and $C_p(n,m)$ are as defined in Section 1.

(ii) If $m>1$, then $x_i = \sigma u_i$ and $y_{i,a} = \sigma v_{i,a}$ where $u_i, v_{i,a} \in H_*(\Omega^3(SU(n)/SU(m));\mathbb{Z}/p)$. If $m=1$, then x_i and $y_{i,a}$ are the image of σu_i and $\sigma v_{i,a}$ under the natural inclusion $\Omega^2(SU(n)<3>) \to \Omega^2 SU(n)$ where $u_i, v_{i,a} \in H_*(\Omega^3(SU(n)<3>);\mathbb{Z}/p)$.

The module of primitives in $H_*(\Omega^2(SU(n)/SU(m));\mathbb{Z}/p)$ is generated by the following elements. We consider the cases $p=2$ and $p>2$ separately. The dimensions of the generators are given for the convenience of the reader.

Let $p=2$.

$$|Q_1^a x_i| = 2^{a+1}i-1 \qquad a\geq 0\ ,\quad i\in A_2(n,m)\cup B_2(n,m)$$

$$|(Q_1^a x_i)^{2^{b+1}}| = 2^{b+1}(2^{a+1}i-1) \qquad a,b\geq 0\ ,\quad i\in B_2(n,m)$$

$$|(y_{i,a})^{2^b}| = 2^b(2^{a+2}i-2) \qquad a,b\geq 0\ ,\quad i\in C_2(n,m)\ .$$

Let $p>2$.

$$|Q^a_{(p-1)}x_i| = 2p^a i-1 \qquad a\geq 0\ ,\quad i\in A_p(n,m)\cup B_p(n,m)$$

$$|(\beta Q_1^{a+1}x_i)^{p^b}| = p^b(2p^{a+1}i-2) \qquad a,b\geq 0\ ,\quad i\in B_p(n,m)$$

$$|(y_{i,a})^{p^b}| = p^b(2p^{a+1}i-2) \qquad a,b\geq 0\ ,\quad i\in C_p(n,m)\ .$$

By inspection of the dimensions in which these generators lie, we obtain the following corollary.

Corollary 8.2. The module of primitives in $H_*(\Omega^2(SU(n)/SU(m));\mathbb{Z}/p)$ has at most one generator in any given dimension.

Corollary 8.3. Let $n>1$. Then in $H_*(\Omega^2 SU(n);\mathbb{Z}/p)$

$$Q^a_{2(p-1)}y_{i,0} = \varepsilon y_{i,a} \quad \text{for some} \quad \varepsilon \not\equiv 0 \bmod p$$

$$Q_1 Q_2^a y_{i,0} = 0 \quad \text{if} \quad p=2$$

$$\beta Q_2^{a+1} y_{i,0} = 0 \qquad \text{if} \quad p>2 .$$

Proof: By Lemma 5.2, in $H_*(\Omega^3(SU(n)\langle 3\rangle);\mathbb{Z}/p)$, $Q^a_{3(p-1)} v_{i,0} = \varepsilon v_{i,a} +$ (decomposables) where $\varepsilon \not\equiv 0 \bmod p$. Thus by Proposition 8.1(ii) the formula $Q^a_{2(p-1)} y_{i,0} = \varepsilon y_{i,a}$ holds. The other formulas hold because the module of primitives in the appropriate dimension is zero. □

Corollary 8.4. Let $j_{n,m}$: $SU(n) \to SU(n)/SU(m)$ be the natural quotient map. Then $(\Omega^2 j_{n,m})_*$ is given by the following formulas.

(i) $$(\Omega^2 j_{n,m})_* \, x_i = \begin{cases} x_i & i \ge m \\ 0 & i<m \end{cases}$$

(ii) $$(\Omega^2 j_{n,m})_* \, y_{i,a} = \begin{cases} y_{i,a} & i \ge m \\ 0 & i<m \end{cases} .$$

Proof: For $n>p$, Corollary 8.4 follows from Proposition 7.1(ii) and Proposition 8.1(ii). If $n \le p$, then localized at p, SU(n) is homotopy equivalent to $S^3 \times \cdots \times S^{2n-1}$ and $SU(n)/SU(m)$ is homotopy equivalent to $S^{2n+1} \times \cdots \times S^{2n-1}$ and $j_{n,m}$ is the obvious projection map. This implies Corollary 8.4 for $n \le p$. □

Theorem 1.5 follows from Proposition 8.1(i) and Corollary 8.3. Proposition 1.15 follows from Corollary 8.4.

Proposition 8.1 is proved by induction. The following lemma provides most of the information needed in the induction step. We assume Proposition 8.1 for $1 \le m < n$. Let $\gamma_{n,m}\colon \Omega S^{2n+1} \to SU(n)/SU(m)$ be a choice of map induced by the fibration $p\colon SU(n+1)/SU(m) \to S^{2n+1}$.

<u>Lemma 8.5.</u> Let $2 \le m < n$.

(i) If $n \not\equiv 0 \bmod p$ or $m > [n/p]$, then $\mathrm{Im}(\Omega^2\gamma_{n,m})_* = 0$ and $\langle \sigma(\ker(\Omega^2\gamma_{n,m})_*) = S(X_n,2)$.

(ii) If $n \equiv 0 \bmod p$ and $m \le [n/p]$, then $\mathrm{Im}(\Omega^2\gamma_{n,m})_* = S(Y_{n/p},1)$ and $\langle \sigma(\ker(\Omega^2\gamma_{n,m})_*)\rangle = S(Y_n,1)$.

<u>Proof:</u> The map $\gamma_{n,m}$ factors as:

$$\begin{array}{ccc} \Omega S^{2n+1} & \xrightarrow{\gamma_{n,m}} & SU(n)/SU(m) \\ & {\scriptstyle \Omega\alpha_n}\searrow \quad \nearrow{\scriptstyle j_{n,m}} & \\ & SU(n) & \end{array}$$

Corollary 8.4 gives $(\Omega^2 j_{n,m})_*$. We compute $(\Omega^3\alpha_n)_*$.

Let $n \not\equiv 0 \bmod p$. By Proposition 8.1(i), $PH_{2n-2}(\Omega^2 SU(n);\mathbb{Z}/p) = 0$. Thus $(\Omega^3\alpha_n)_* z_{n,3} = 0$ where $z_{n,3}$ is a generator of $H_{2n-2}(\Omega^3 S^{2n+1};\mathbb{Z}/p)$. Hence $(\Omega^3\alpha_n)_* = 0$.

Let $n \equiv 0 \bmod p$. By Proposition 8.1(i), $PH_{2n-2}(\Omega^2 SU(n);\mathbb{Z}/p)$ is generated by $y_{n/p,0}$. By Lemma 3.1, $(\Omega^3\alpha_n)_* z_{n,3} \ne 0$. Thus $(\Omega^3\alpha_n)_* z_{n,3} = \lambda y_{n/p,0}$ for some $\lambda \not\equiv 0 \bmod p$. Using Corollary 8.3, this completely determines the map $(\Omega^3\alpha_n)_*$.

Lemma 8.5 follows directly from the above remarks. □

Proof of Proposition 8.1: Proceed by induction on n. If $n=2$, then $m=1$ and $SU(n)/SU(m) = S^3$. Proposition 8.1 follows from Theorem 4.4.

Assume Proposition 8.1 for n. Proposition 8.1 for n+1 follows by applying Proposition 4.6, along with Lemma 8.5, to the fibration

$$SU(n)/SU(m) \xrightarrow{i} SU(n+1)/SU(m) \xrightarrow{p} S^{2n+1} .$$

Note when $n \not\equiv 0 \bmod p$ or $m > [n/p]$, $\langle\sigma(\ker(\Omega^2\gamma_{n,m})_*)\rangle = S(X_n,2)$ and when $n \equiv 0 \bmod p$ and $m \leq [n/p]$, $\langle\sigma(\ker(\Omega^2\gamma_{n,m})_*)\rangle = S(Y_n,1)$. Both X_n and Y_n are in the image of the suspension of $\Omega^3 p_*(H_*(\Omega^3(SU(n)/SU(m));\mathbb{Z}/p)$. Thus Proposition 4.6 gives $H_*(\Omega^2(SU(n)/SU(m));\mathbb{Z}/p)$ as a Hopf-algebra, and that x_i and $y_{i,a}$ may be chosen as in Proposition 8.1(ii). □

Proof of Proposition 1.13. The method used to prove that λ_1 is trivial in $H_*(\Omega^2(SU(n)/SU(m));\mathbb{Z}/2)$ and that λ_2 is trivial in $H_*(\Omega^2 SU(n);\mathbb{Z}/2)$ is as follows.

Suppose that $\lambda_n(a,b) \neq 0$ for some $a,b \in H_*(\Omega^{n+1}X;\mathbb{Z}/2)$. Thus there exist a and b in $H_*(\Omega^{n+1}X;\mathbb{Z}/2)$ such that $\lambda_n(a,b) \neq 0$ and if $\lambda_n(a',b') \neq 0$ for some $a',b' \in H_*(\Omega^{n+1}X;\mathbb{Z}/2)$, then $|a| \leq |a'|,|b'|$ and $|b| \leq \max\{|a'|,|b'|\}$. We say that the pair a,b is minimal.

Note that $Sq_*^j \lambda_n(a,b) = \sum_{i=0}^{j} \lambda_n(Sq_*^i a, Sq_*^{j-i} b)$ which is zero if a,b is minimal and $j>0$.

First we prove that λ_2 is trivial in $H_*(\Omega^2 SU(n);\mathbb{Z}/p)$. Proceed by contradiction. Assume $\lambda_2 \neq 0$. By our comments above on minimal pairs, one of the following cases must occur:

(i) $\lambda_2(x_i, x_j) \neq 0$, $Sq_*^k(\lambda_2(x_i, x_j)) = 0$ for $k>0$

(ii) $\lambda_2(x_i, y_{j,0}) \neq 0$, $Sq_*^k(\lambda_2(x_i, y_{j,0})) = 0$ for $k>0$

(iii) $\lambda_2(y_{i,0}, y_{j,0}) \neq 0$, $Sq_*^k(\lambda_2(y_{i,0}, y_{j,0})) = 0$ for $k>0$.

The first two cases are impossible because the module of primitives in $H_*(\Omega^2 SU(n);\mathbb{Z}/2)$ in the appropriate dimensions is zero. In the third case we must have that $\lambda_2(y_{i,0}, y_{j,0})$ is either $(Q_{(p-1)}^2 x_k)^2$ or $Q_2 y_k$ for some k. But Sq_*^2 on these elements is non-zero. Thus none of the cases may occur. Hence λ_2 is trivial.

Since λ_1 is trivial on the image of $H_*(\Omega^2 SU(n);\mathbb{Z}/2)$ in $H_*(\Omega^2(SU(n)/SU(m));\mathbb{Z}/2)$, if λ_1 in $H_*(\Omega^2(SU(n)/SU(m));\mathbb{Z}/2)$ were non-trivial, then $\lambda_1(x_i, x_j) \neq 0$ for some i,j where both i and j are even. But the module of primitives in this dimension is zero. Thus $\lambda_1 = 0$. □

<u>Proof of Proposition 1.12.</u> By Corollary 8.3, to prove the formulas given in Proposition 1.12, it suffices to prove

that $Q_2 x_i = 0$ in $H_*(\Omega^2 SU(n);\mathbb{Z}/2)$. By considering the dimensions in which the primitives in $H_*(\Omega^2 SU(n);\mathbb{Z}/2)$ lie, $Q_2 x_i = 0$ in all cases except when $i = n-1$. Then

$$Q_2 x_{n-1} = \begin{cases} \varepsilon(x_k)^4 & n = 2k,\ k \not\equiv 0 \bmod 2 \\ \varepsilon(y_{k,0})^2 & n = 2k,\ k \equiv 0 \bmod 2\ . \end{cases}$$

But in $H_*(\Omega^3(SU(n)\langle 3\rangle);\mathbb{Z}/2)$, $Sq_*^1 Q_1^2 u_k = (Q_1 u_k)^2$ and $Sq_*^1(Q_1 v_{k,0}) = v_{k,0}^2$, whereas $Sq_*^1 Q_3 u_{n-1} = \lambda_3(u_{n-1}, Sq_*^1 u_{n-1}) = 0$ since $Sq_*^1 u_{n-1} = 0$ for dimension reasons. Thus $Q_3 u_{n-1}$ is not equal to either $Q_1^2 u_k$ or $Q_1 v_{k,0}$. By Proposition 8.1(ii), $Q_2 x_{n-1}$ is not equal to either x_k^4 or $y_{k,0}^2$. □

§9. STEENROD OPERATIONS IN $H_*(\Omega^2(SU(n)/SU(m));\mathbb{Z}/2)$ AND PROOFS OF PROPOSITIONS 1.14 AND 1.16

To compute the Steenrod operations on the elements x_i in $H_*(\Omega^2(SU(n)/SU(m));\mathbb{Z}/2)$, we suspend to $H_*(\Omega(SU(n)/SU(m));\mathbb{Z}/2)$ where the Steenrod operations are known. The structure of $H_*(\Omega(SU(n)/SU(m));\mathbb{Z}/2)$ as a Hopf algebra over the Steenrod algebra is given in the following results [8].

<u>Theorem 9.1.</u> $H_*(\Omega SU(n);\mathbb{Z}/2)$ is isomorphic to $P[w_1,w_2,\ldots,w_{n-1}]$ where $|w_i| = 2i$. The coproduct structure is given by

$$\Delta w_i = \sum_{j=0}^{i} w_j \otimes w_{i-j} \qquad \text{where} \quad w_0=1 .$$

The action of the Steenrod algebra is given by

$$Sq_*^{2j} w_i = (i-2j,j)w_{i-j} .$$

It is also shown in [8] that a basis for the module of primitives in dimension $\leq 2n-2$ of $H_*(\Omega SU(n);\mathbb{Z}/2)$ is given by the Newton polynomials, which we denote by $\bar{w}_i$, for $1\leq i\leq n-1$. $\bar{w}_i$ is defined inductively by

$$\bar{w}_1 = w_1 \quad \text{and} \quad \bar{w}_i = iw_i - \sum_{j=1}^{i-1} w_j\bar{w}_{i-j} .$$

Also

$$Sq_*^{2j}\bar{w}_i = (i-2j-1,j)\bar{w}_{i-j} .$$

Notice that the next two propositions follow immediately from Theorem 9.1.

<u>Proposition 9.2.</u> $H_*(\Omega(SU(n)/SU(m));\mathbb{Z}/2)$ is isomorphic to $P[w_m,\ldots,w_{n-1}]$ where $|w_i| = 2i$. Furthermore, $\Omega i_*: P[w_1,\ldots,w_{n-1}] \to P[w_m,\ldots,w_{n-1}]$ is the obvious quotient map, where $i: SU(n) \to SU(n)/SU(m)$.

<u>Proposition 9.3.</u> The image of $w_{n-1} \in H_*(\Omega(SU(n)/SU(m));\mathbb{Z}/2)$ under the map Ωp_*, where $p: SU(n)/SU(m) \to S^{2n-1}$, is the generator $H_{2n-2}(\Omega S^{2n-1};\mathbb{Z}/2)$, which we denote by $z_{n-1,1}$ as in Theorem 4.4.

<u>Lemma 9.4.</u> (i) Consider $x_i \in H_*(\Omega^2 SU(n);\mathbb{Z}/2)$ for $i \not\equiv 0 \bmod 2$. Then $\sigma x_i = \bar{w}_i$.

(ii) Consider $x_i \in H_*(\Omega^2(SU(n)/SU(m));\mathbb{Z}/2)$ for $i \equiv 0 \bmod 2$. Then $\sigma x_i = w_i$.

<u>Proof:</u> To show either (i) or (ii), it suffices to consider the case $i = n-1$. The element $x_i \in H_*(\Omega^2(SU(n)/SU(m));\mathbb{Z}/2)$ was chosen so that its image under $\Omega^2 p_*$ is the generator of $H_{2n-3}(\Omega^2 S^{2n-1};\mathbb{Z}/2)$. Thus σx_i is a primitive whose image in $H_{2n-2}(\Omega S^{2n-1};\mathbb{Z}/2)$ is $z_{n-1,1}$. Both cases of Lemma 9.4 now follow by inspection of the module of primitives in $H_{2n-2}(\Omega(SU(n)/SU(m));\mathbb{Z}/2)$. □

<u>Proof of Proposition 1.16.</u> Let $i \equiv 0 \bmod 2$. Note that the module of primitives $H_{2k}(\Omega^2(SU(n)/SU(m));\mathbb{Z}/2)$ is zero

if $k \le n-1$. Thus $Sq_*^{2j+1} x_i = 0$. The module of primitives in $H_{2k-1}(\Omega^2(SU(n)/SU(m));\mathbb{Z}/2)$ is generated by x_k if $m \le k \le i$. Thus

$$Sq_*^{2j} x_k = \begin{cases} \varepsilon x_{i-j} & \text{if } i-j \ge m \\ 0 & \text{if } i-j < m . \end{cases}$$

To determine ε, suspend to $H_*(\Omega(SU(n)/SU(m));\mathbb{Z}/2)$ and use the formula giving the action of the Steenrod algebra found in Theorem 9.1. □

<u>Proof of Proposition 1.14.</u> To prove Proposition 1.14(i), we proceed as in the proof of Proposition 1.16. Since $H_{2\ell}(\Omega^2 SU(n);\mathbb{Z}/2)$ is zero if $\ell \le n-1$, $Sq_*^{2j+1} x_i = 0$. The module of primitives in $H_{2\ell-1}(\Omega^2 SU(n);\mathbb{Z}/2)$ is generated by $Q_1^a x_k$ where $\ell = 2^a k$, $k \not\equiv 0 \bmod 2$, for $1 \le \ell \le n-1$. Thus

$$Sq_*^{2j} x_i = \varepsilon Q_1^a x_k \quad \text{where} \quad i-j = 2^a k,\ k \not\equiv 0 \bmod 2 .$$

To determine ε, suspend to $H_*(\Omega SU(n);\mathbb{Z}/p)$ and use the formula giving the Steenrod operations found in the remarks following Theorem 9.1. Note that $\sigma(Q_1^a x_k) = Q_0^a \bar{w}_k = (\bar{w}_k)^{2^a} = \bar{w}_{2^a k}$.

To prove Proposition 1.14(ii), it suffices to show that in $H_*(\Omega^2 SU(n);\mathbb{Z}/2)$ for $n \equiv 1 \bmod 2$, that

$$Sq_*^j y_{n-1,0} = 0 \quad \text{for} \quad j \not\equiv 0 \bmod 4$$

$$Sq_*^{4j} y_{n-1,0} = \begin{cases} (i-2j,j)\, y_{n-1-j,0} & n-1-j \equiv 0 \bmod 2,\ n-1-j > \left[\frac{n-1}{2}\right] \\ (i-2j,j)\,(x_{n-1-j})^2 & n-1-j \equiv 0 \bmod 2,\ n-1-j > \left[\frac{n-1}{2}\right] \\ 0 & \text{otherwise} . \end{cases}$$

Let $i\colon SU(n) \to SU(2n-2)$ be the natural inclusion map. By Lemma 8.5, $\Omega^2 i_*(y_{n-1,0})$ is the image of the generator of $H_{4n-6}(\Omega^3 S^{4n-3};\mathbb{Z}/2)$. Thus all Steenrod operations on $\Omega^2 i_*(y_{n-1,0})$ are zero. Since $\Omega^2 i_*$ applied to an odd-dimensional primitive is non-zero, $Sq_*^{2j+1} y_{n-1,0} = 0$ if and only if $Sq_*^{2j+1}(\Omega^2 i_* y_{n-1,0}) = 0$. Thus $Sq_*^{2j+1} y_{n-1,0} = 0$.

Let $j\colon SU(n) \to SU(n)/SU((n+1)/2)$ be the natural quotient map. Note that $\Omega^2 j_*$ applied to an even-dimensional primitive is non-zero. Thus to compute $Sq_*^{2j} y_{n-1,0}$, it suffices to compute $Sq_*^{2j}(\Omega^2 j_* y_{n-1,0})$. By Proposition 1.15, $\Omega^2 j_* y_{n-1,0} = (x_{(n-1)})^2$. But the Steenrod operations on $(x_{(n-1)})^2$ are given by Propositions 1.16. □

§10. THE BOCKSTEIN SPECTRAL SEQUENCE FOR $H_*(\Omega^2(SU(n)/SU(m));\mathbb{Z}/p)$ AND THE PROOF OF PROPOSITION 1.17

Recall that the Bockstein spectral sequence for a space X is obtained from the exact couple

$$\begin{array}{ccc} H_*(X;\mathbb{Z}) & \xrightarrow{\quad p \quad} & H_*(X;\mathbb{Z}) \\ {\scriptstyle d}\nwarrow & & \swarrow{\scriptstyle \rho} \\ & H_*(X;\mathbb{Z}/2) & \end{array}$$

where p is multiplication by p, ρ is reduction mod-p and d is the boundary homomorphism in the long exact sequence.

The r^{th} differential in the spectral sequence is denoted β^r. We will abuse notation and write $\beta^r x = y$ for $x, y \in H_*(X;\mathbb{Z}/p)$. $\beta^r x = y$ can be interpreted as follows: There exists $z \in H_*(X;\mathbb{Z})$ such that $dx = p^{r-1}z$. y is the mod-p reduction of z.

To compute the differentials in the Bockstein spectral sequence for $H_*(\Omega^2(SU(n)/SU(m));\mathbb{Z}/p)$, we need the following lemma which is a homology analogue to the Bockstein lemma in [10].

<u>Lemma 10.1.</u> Let $F \xrightarrow{i} X \xrightarrow{p} B$ be an orientable fibration. Let $u \in H_{n+1}(B;\mathbb{Z}/p)$ and $v \in H_{n+1}(F;\mathbb{Z}/p)$ be such that $\beta^s v = 0$ for $1 \le s < r$ and in the Serre spectral sequence, u transgresses to $\beta^r v$. Then $\beta^s(i_* v) = 0$ for $1 \le s \le r$ and thus $\beta^{r+1}(i_* v)$ is defined. Furthermore, $p_*(\beta^{r+1}(i_* v)) = \beta^1 u$ up to indeterminacy given by $p_*(\beta^1(H_{n+1}(X;\mathbb{Z}/p)))$.

Proof: Since u transgresses to $\beta^r v$, there exists an element w in $H_{n+1}(X,F;\mathbb{Z}/p)$ such that $\beta^r v = \partial w$ where ∂ is the boundary homomorphism in the long exact homology sequence for a pair, and $p_*(w+w')=u$ for some $w' \in H_{n+1}(X;\mathbb{Z}/p)$. Since $\partial w = \beta^r v$, there exist a z in $H_n(F;\mathbb{Z})$ such that $dv = p^{r-1}z$ and ∂w is the mod-p reduction of z.

Consider this on the chain level. Let $a \in C_{n+1}X$ be such that the class of a represents w. Since ∂w is the mod-p reduction of z, there exists $b \in C_n F$ such that the class of b represents z and $\partial a = b-2c$ for some $c \in C_n X$. Let $[\cdot]$ denote the class of a cycle. Since $d(i_* v) = 2^{r-1}[b] = 2^{r-1}[2c+\partial a] = 2^r[c]$, $\beta^{r+1}(i_* v) = [c]$. Also $d(p_* w) = [c]$, so $\beta^1(p_* w) = p_*([c])$. Thus $p_*(\beta^{r+1}(i_* v)) = \beta^1(p_*(w)) = \beta^1 u - \beta^1(p_*(w')) . \square$

Proof of Proposition 1.17: The differentials in the Bockstein spectral sequence for $H_*(\Omega^2(SU(n)/SU(m));\mathbb{Z}/p)$ are given by Proposition 1.17. Recall that the Bockstein spectral sequence is a spectral sequence of Hopf algebras [2]. We prove Proposition 1.17 by induction on n. Fix $m \geq 1$. Let $n = m+1$. By Theorem 1.10, $H_*(\Omega^2(SU(n)/SU(m));\mathbb{Z}/p)$ is isomorphic to one of the following Hopf algebras:

(i) Let p=2.

$$P[Q_1^a x_{n-1} \mid a \geq 0]$$

(ii) Let $p>2$.

$$E[Q^a_{(p-1)}x_{n-1} \mid a\geq 0] \otimes P[\beta Q^a_{(p-1)}x_{n-1} \mid a>0]$$

Clearly, for all p, $\beta^1(Q^a_{(p-1)}x_{n-1}) = \beta Q^a_{(p-1)}x_{n-1}$. When $p=2$, $Sq^1_* = \beta$. Thus $\beta Q^a_{(p-1)}x_{n-1} = (Q^{a-1}_{(p-1)}x_{n-1})^2$. Hence the E^2 for the Bockstein spectral sequence for $H_*(\Omega^2(SU(n)/SU(m));\mathbb{Z}/p)$ is $E[x_{n-1}]$. Thus there can be no higher differentials. So we have shown that Proposition 1.17 holds for m+1.

Assume that the differentials in the Bockstein spectral sequence for $H_*(\Omega^2(SU(n)/SU(m));\mathbb{Z}/p)$ are given as in Proposition 1.17. We will show by induction on r that the E^r term for the Bockstein spectral sequence of $H_*(\Omega^2(SU(n+1)/SU(m));\mathbb{Z}/p)$ as a Hopf algebra is given by the following:

When $r=1$,

$$E^1 = H_*(\Omega^2(SU(n+1)/SU(m));\mathbb{Z}/p) \ .$$

When $r>1$,

$$\begin{aligned} E^r = {} & E[Q^a_{(p-1)}x_i \mid 0\leq a<s_{n+1}(i),\ i\in A_p(n+1,m) \cup B_p(n+1,m)] \\ & \otimes E[Q^a_{(p-1)}x_i \mid a \geq s_{n+1}(i) \geq r,\ i\in A_p(n+1,m)] \\ & \otimes P[y_{p^{s-1}i,a} \mid a\geq 0,\ s\geq r,\ i\in A_p(n+1,m), \\ & \qquad p^{s-1}i\in C_p(n+1,m)] . \end{aligned}$$

For $r=1$, the E^1 term is clearly given as above. Assume that the E^r term is given as above. Note that

$s_n(k) = s_{n+1}(k)$ except when $n = p^a k$. In this case, $s_n(k)+1 = s_{n+1}(k)$. The following formulas give β^r.

(i) $\beta^1(Q^a_{(p-1)}x_i) = \beta Q^a_{(p-1)}x_i$ where $a \geq 1$, $i \in B_p(n+1,m)$. The above is simply a tautology. Note, however, that for a and i in the above range $\beta Q^a_{(p-1)}x_i \neq 0$, and when $p=2$, $\beta Q^a_{(p-1)}x_i = (Q_{(p-1)}x_i)^2$.

(ii) $\beta^r(Q^a_{(p-1)}x_i) = 0$ where $0 \leq a < s_{n+1}(i)$, $i \in A_p(n+1,m) \cup B(n+1,m)$. This follows because $|\beta^r(Q^a_{(p-1)}x_i)| = 2p^a i - 2 < 2n$, and because there are no even-dimensional primitives of dimension less than 2n in E^r.

(iii) $\beta^r(y_{p^{s-1}i,a}) = 0$ where $a \geq 0$, $s \geq r$, $i \in A_p(n+1,m)$, $p^{s-1}i \in C_p(n+1,m)$. The element $y_{p^{s-1}i,a}$ is in the image of $H_*(\Omega^2(SU(n)/SU(m);\mathbb{Z}/p)$ except when $n = p^{s-1}i$. If $n \neq p^{s-1}i$, then $\beta^r y_{p^{s-1}i,a} = 0$ since this is true in $H_*(\Omega^2(SU(n)/SU(m);\mathbb{Z}/p)$. If $n = p^{s-1}i$, then $\beta^r y_{p^{s-1}i,a} = 0$, since there are no primitives in E^r of dimension $2p^{a+1}n-3$.

(iv) $\beta^r(Q^a_{(p-1)}x_i) = 0$ where $a \geq s_{n+1}(i) > r$, $i \in A_p(n+1,m)$. The element $Q^a_{(p-1)}x_i$ is in the image of $H_*(\Omega^2(SU(n)/SU(m));\mathbb{Z}/p)$. If $n \neq p^r i$, the $s_{n+1}(i) > r$ implies $s_n(i) > r$. Thus $\beta^r(Q^a_{(p-1)}x_i) = 0$ because this is true in $H_*(\Omega^2(SU(n)/SU(m));\mathbb{Z}/p)$. If $n = p^r i$, then in $H_*(\Omega^2(SU(n)/SU(m));\mathbb{Z}/p)$, $\beta^r(Q^a_{(p-1)}x_i) = y_{p^{r-1}i,a-r}$.

But the image of $y_{p^{r-1}i,a-r}$ in $H_*(\Omega^2(SU(n+1)/SU(m));\mathbb{Z}/p)$ is zero. Thus $\beta^r(Q^a_{(p-1)}x_i) = 0$ in $H_*(\Omega^2(SU(n+1)/SU(m));\mathbb{Z}/p)$.

(v) $\beta^r(Q^a_{(p-1)}x_i) = y_{p^{r-1}i,a-r}$ where $a \geq s_{n+1}(i) = r$, $i \in A_p(n+1,m)$. The element $Q^a_{(p-1)}x_i$ is in the image of $H_*(\Omega^2(SU(n)/SU(m));\mathbb{Z}/p)$. If $n \neq p^{r-1}i$, then $s_{n+1}(i) = r$ implies $s_n(i) = r$. Thus $\beta^r(Q^a_{(p-1)}x_i) = y_{p^{r-1}i,a-r}$ since this is true in $H_*(\Omega^2(SU(n)/SU(m));\mathbb{Z}/p)$. If $n = p^{r-1}i$, then in the Serre spectral sequence for the fibration

$$\Omega^2(SU(n)/SU(m)) \to \Omega^2(SU(n+1)/SU(m)) \to \Omega^2S^{2n+1} .$$

The element $Q^a_{(p-1)}z_{n,2}$ transgresses to the element $y_{p^{r-2}i,a}$, where $z_{n,2}$ is a choice of generator for $H_{2n-1}(\Omega^2S^{2n+1};\mathbb{Z}/p)$. This follows from Corollary 4.12 and Lemma 8.5. In $H_*(\Omega^2(SU(n)/SU(m));\mathbb{Z}/p)$, $\beta^{r-1}(Q^a_{(p-1)}x_i) = y_{p^{r-2}i,a-r+1}$. Thus by Lemma 10.1, $\beta^r(Q^a_{(p-1)}x_i)$ is represented by an element in $H_*(\Omega^2(SU(n+1)/SU(n));\mathbb{Z}/p)$ whose image in $H_*(\Omega^2S^{2n+1};\mathbb{Z}/p)$ is $\beta Q^{a-r+1}_{(p-1)}z_{n,2}$ up to indeterminacy. But in this case, the indeterminacy is zero because the image of $H_*(\Omega^2(SU(n+1)/SU(m));\mathbb{Z}/p)$ in $H_*(\Omega^2S^{2n+1};\mathbb{Z}/p)$ is generated by elements of the form $\beta Q^a_{(p-1)}z_{n,2}$. Since $y_{p^{r-1}i,a-r}$ is an element of $H_*(\Omega^2(SU(n+1)/SU(n));\mathbb{Z}/p)$ whose image in

$H_*(\Omega^2 S^{2n+1};\mathbb{Z}/p)$ is $\beta Q^{a-r+1}_{(p-1)} z_{n,2}$ and $y_{p^{r-1}i,a-r}$ generates the module of primitives in dimension $2p^a i-2$ of E^r, we must have that $\beta(Q^a_{(p-1)} x_i) = y_{p^{r-1}i,a-r}$.

By (i) through (v), the E^{r+1} term for the Bockstein spectral sequence of $H_*(\Omega^2(SU(n+1)/SU(n));\mathbb{Z}/p)$ is given as above. This implies that the differential in the Bockstein spectral sequence for $H_*(\Omega^2(SU(n+1)/SU(n));\mathbb{Z}/p)$ are given by Proposition 1.17. □

Corollary 1.18 and Example 1.19 follow by inspection of the differentials in the appropriate Bockstein spectral sequence.

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